Topological superconductivity in low dimensions

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Comprendre le monde, construire l'avenir®

Some bibliography

Good reviews:

- J. Alicea, Rep. Prog. Phys. 75, 076501, (2012).
- Elliot&Franz, Rev. Mod. Phys. 87, 137 (2015).
- TS parts in Hasan and Kane, Rev. Mod. Phys.82, 3045, (2010) and Qi and Zhang, Rev. Mod. Phys., 83, 1057 (2011).
- Chapters 16-18 in “Topological Insulators and Superconductors” Bernevig and Hughes, Princeton Univ. Press. (2013).
- Chapters 9 &10 in “Topological Insulators: Dirac equation on condensed matter”, S.-Q.Shen, Springer-Verlag 2012.

Introduction

- Soon after their discovery, the study of time-reversal invariant topological insulators was generalized to time-reversal invariant topological superconductors and superfluids. Even before TIs, the study of topological phases in superconductors and superfluids had a long history (see Volovik, “The Universe in a Helium droplet”, Clarendon, 2003).

- Key concept: there is a direct analogy between superconductors and insulators because the Bogoliubov-de Gennes (BdG) Hamiltonian that describes quasiparticle excitations in a superconductor is analogous to the Hamiltonian of a band insulator, with the superconducting gap corresponding to the insulating gap.

- Key property: gapless surface states of topological superconductors have Majorana character.

What is a Majorana fermion ?

$$\boxed{(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\,\Psi = 0}$$

Majorana's question (1937): does the Dirac equation necessarily involve complex fields?

What is a Majorana fermion ?

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\,\Psi = 0}$$

Majorana's answer: **No**, if Weyl matrices are purely imaginary

$$\gamma^0 = \sigma_y \otimes \sigma_x$$

$$\gamma^1 = i\sigma_x \otimes 1$$

$$\gamma^2 = i\sigma_z \otimes 1$$

$$\gamma^3 = i\sigma_y \otimes \sigma_y$$

$$\Rightarrow \quad \Psi = \Psi^*$$

- Neutral particle equals its own antiparticle.
- Very relevant in neutrino physics.
- Many experimental efforts to search for Majorana neutrinos are underway.

Majorana, Nuovo Cimento **14**, 171 (1937)

What is a Majorana fermion ?

RELEVANT FOR US: in the most general case, it is sufficient to demand in any representation that there exists a matrix Ξ such that

$$\Psi = \Xi \Psi^* = \Psi^C$$

$$\Xi = \tau_y \otimes \sigma_y = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$$

is the charge conjugation operator

perspective

NATURE PHYSICS | VOL 5 | SEPTEMBER 2009 | www.nature.com/naturephysics

Majorana returns

Frank Wilczek

www.sciencemag.org SCIENCE VOL 332 8 APRIL 2011

Published by AAAS

Search for Majorana Fermions Nearing Success at Last?

Researchers think they are on the verge of discovering weird new particles that borrow a trick from superconductors and could give a big boost to quantum computers

Physics

Physics **3**, 24 (2010)

Viewpoint

Race for Majorana fermions

Marcel Franz

Department of Physics and Astronomy, University of British Columbia, Va

Published March 15, 2010

The race for realizing Majorana fermions—elusive particles th
we still await ideal materials to work with.

Physics

Physics 4, 67 (2011)

Viewpoint

Majorana fermions inch closer to reality

Taylor L. Hughes

University of Illinois at Urbana-Champaign, 1110 W. Green St., Urbana, IL 61801, USA

Published August 22, 2011

MAJORANA FERMIONS IN CONDENSED MATTER?

ALL PROPOSALS ARE BASED ON
HYBRID SYSTEMS INVOLVING
SUPERCONDUCTORS

Outline

0) Bogoliubov-de Gennes formalism

I) 1D topological superconductivity

- The Kitaev spinless model
- A few physical realizations

II) 2D topological superconductivity

- Spinless 2D topological superconductor
- Spinful case: helical superconductor

0) Introduction:
BdG formalism

Bogoliubov-de Gennes formalism

Bogoliubov - de Gennes (BdG) formalism of superconductivity: essentially BCS theory adapted to describe quasiparticle excitations in superconductors.

$$H = \sum_{\mathbf{p},\sigma} c^\dagger_{\mathbf{p}\sigma} \left(\frac{p^2}{2m} - \mu \right) c_{\mathbf{p}\sigma} \equiv \sum_{\mathbf{p},\sigma} c^\dagger_{\mathbf{p}\sigma} \epsilon(p) c_{\mathbf{p}\sigma}$$

Ground state: $$|\Omega\rangle = \prod_{\mathbf{p}:\, \epsilon(p)<0} \prod_{\sigma} c^\dagger_{\mathbf{p}\sigma} |0\rangle$$

Use fermionic anti-commutation relations:

$$H = \frac{1}{2} \sum_{\mathbf{p}\sigma} \left[c^\dagger_{\mathbf{p}\sigma} \epsilon(p) c_{\mathbf{p}\sigma} - c_{\mathbf{p}\sigma} \epsilon(p) c^\dagger_{\mathbf{p}\sigma} \right] + \frac{1}{2} \sum_{\mathbf{p}} \epsilon(p)$$

$$= \frac{1}{2} \sum_{\mathbf{p}\sigma} \left[c^\dagger_{\mathbf{p}\sigma} \epsilon(p) c_{\mathbf{p}\sigma} - c_{-\mathbf{p}\sigma} \epsilon(-p) c^\dagger_{-\mathbf{p}\sigma} \right] + \frac{1}{2} \sum_{\mathbf{p}} \epsilon(p)$$

Bogoliubov-de Gennes formalism

Bogoliubov - de Gennes (BdG) formalism of superconductivity: essentially BCS theory adapted to describe quasiparticle excitations in superconductors.

TRICK: redundant description to treat electrons and holes at the same footing

Nambu basis $\Psi_{\mathbf{p}} \equiv (c_{\mathbf{p}\uparrow}\ c_{\mathbf{p}\downarrow}\ c^{\dagger}_{-\mathbf{p}\uparrow}\ c^{\dagger}_{-\mathbf{p}\downarrow})^{T}$

$$H = \sum_{\mathbf{p}} \Psi^{\dagger}_{\mathbf{p}} H_{\mathrm{BdG}}(\mathbf{p}) \Psi_{\mathbf{p}} + \text{constant},$$

with
$$H_{\mathrm{BdG}}(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} \epsilon(p) & 0 & 0 & 0 \\ 0 & \epsilon(p) & 0 & 0 \\ 0 & 0 & -\epsilon(-p) & 0 \\ 0 & 0 & 0 & -\epsilon(-p) \end{pmatrix}.$$

Be careful: Different choice of Nambu basis are used by different authors and sometimes may evolve along a paper and/or simply not be specified ….

Particle-hole symmetry

Intrinsic built-in particle-hole redundancy :

$$\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$$

The p/h operator Ξ anticommutes with the Hamiltonian

$$\Xi = \tau^x \otimes \mathbb{1}_2 \, \mathcal{K} \Rightarrow \Xi^2 = +1$$

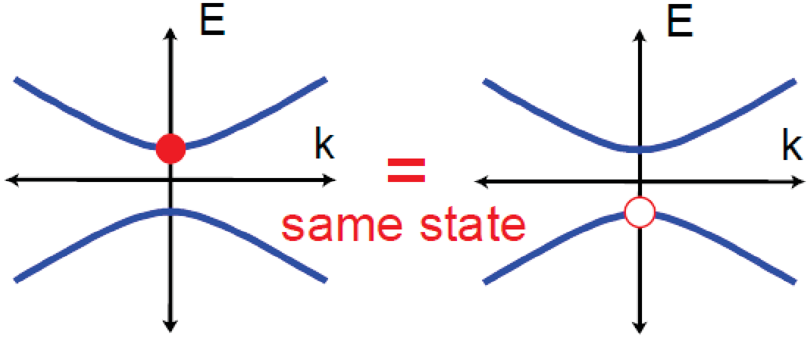
Pauli matrix acts in p/h space

Complex conjugation (antiunitary operator)

If φ_E is an eigenstate with energy +E then

$\varphi_{-E} = \Xi \varphi_E$ is an eigenstate with energy - E

$$\Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



Why such artificial redundency ?

Easier to handle mean field superconductivity

S-wave SC

$$H_\Delta = \Delta c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} + \Delta^* c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow}$$

$$= \frac{1}{2}\left[\Delta\left(c^\dagger_{\mathbf{p}\uparrow} c^\dagger_{-\mathbf{p}\downarrow} - c^\dagger_{-\mathbf{p}\downarrow} c^\dagger_{\mathbf{p}\uparrow}\right) + \Delta^*\left(c_{-\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow} - c_{\mathbf{p}\uparrow} c_{-\mathbf{p}\downarrow}\right)\right]$$

Coupling between particle and hole sectors

$$H + H_\Delta = \sum_{\mathbf{p}} \Psi^\dagger_{\mathbf{p}} H_{\mathrm{BdG}}(\mathbf{p}, \Delta) \Psi_{\mathbf{p}}$$

$$H_{\mathrm{BdG}}(\mathbf{p}, \Delta) = \frac{1}{2}\begin{pmatrix} \epsilon(p) & 0 & 0 & \Delta \\ 0 & \epsilon(p) & -\Delta & 0 \\ 0 & -\Delta^* & -\epsilon(-p) & 0 \\ \Delta^* & 0 & 0 & -\epsilon(-p) \end{pmatrix}$$

Or more compactly

$$H_{\mathrm{BdG}}(\mathbf{p}, \Delta) = \epsilon(p)\tau^z \otimes I_{2\times 2} - (\mathrm{Re}\Delta)\tau^y \otimes \sigma^y - (\mathrm{Im}\Delta)\tau^x \otimes \sigma^y$$

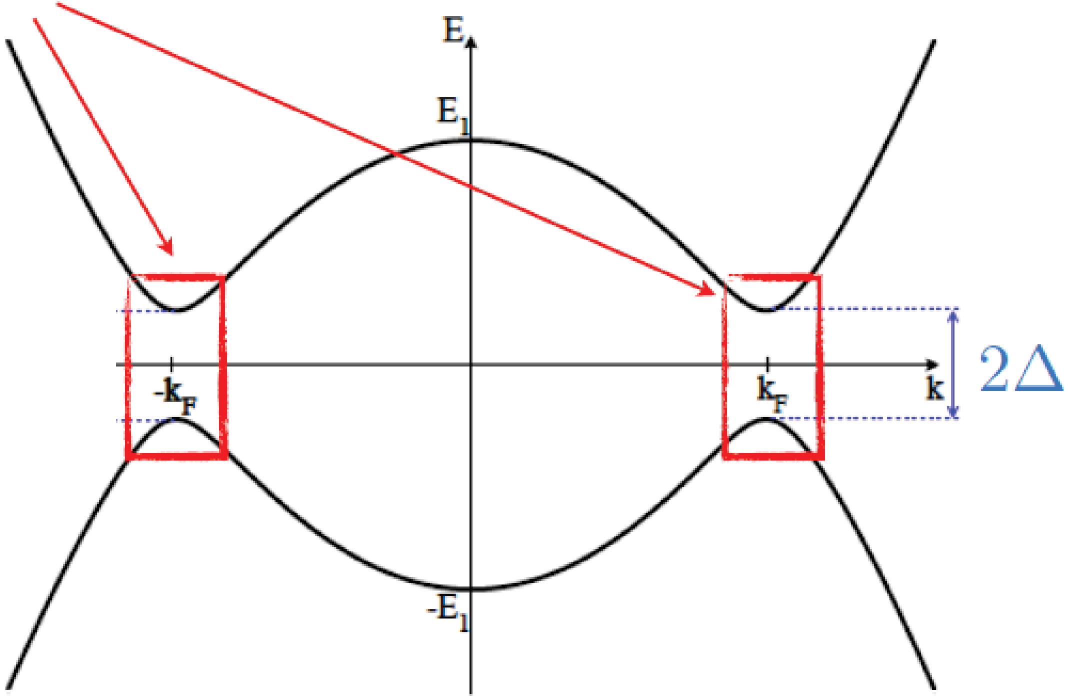
particle-hole Pauli matrix

spin Pauli matrix

BdG Spectrum

$$H_{\mathrm{BdG}}(\mathbf{p},\Delta)=\frac{1}{2}\begin{pmatrix}\epsilon(p) & 0 & \boxed{\begin{matrix}0 & \Delta\\ -\Delta & 0\end{matrix}}\\ 0 & \epsilon(p) & \\ \boxed{\begin{matrix}0 & -\Delta^*\\ \Delta^* & 0\end{matrix}} & & \begin{matrix}-\epsilon(-p) & 0\\ 0 & -\epsilon(-p)\end{matrix}\end{pmatrix} \Rightarrow E_\pm=\pm\sqrt{\epsilon(\mathbf{p})^2+|\Delta|^2}$$

Coupling between particle
and hole sectors



Behaves like an single-particle insulator ⇒ all topological concepts apply

BdG Spectrum

Spectrum is similar to a band insulator with particle-hole symmetry. A key difference, however, is that excitations in the superconductor are superpositions of electrons and holes

$$\gamma^\dagger_{+,\mathbf{p}\uparrow} = e^{i\theta/2} \sin a_{\mathbf{p}}\, c^\dagger_{\mathbf{p}\uparrow} + e^{-i\theta/2} \cos a_{\mathbf{p}}\, c_{-\mathbf{p}\downarrow},$$

$$\gamma^\dagger_{+,\mathbf{p}\downarrow} = -e^{i\theta/2} \sin a_{\mathbf{p}}\, c^\dagger_{\mathbf{p}\downarrow} + e^{-i\theta/2} \cos a_{\mathbf{p}}\, c_{-\mathbf{p}\uparrow}$$

$$\gamma^\dagger_{-,\mathbf{p}\uparrow} = e^{i\theta/2} \sin \beta_{\mathbf{p}}\, c^\dagger_{\mathbf{p}\uparrow} + e^{-i\theta/2} \cos \beta_{\mathbf{p}}\, c_{-\mathbf{p}\downarrow},$$

$$\gamma^\dagger_{-,\mathbf{p}\downarrow} = -e^{i\theta/2} \sin \beta_{\mathbf{p}}\, c^\dagger_{\mathbf{p}\downarrow} + e^{-i\theta/2} \cos \beta_{\mathbf{p}}\, c_{-\mathbf{p}\uparrow}$$

Only two independent excitations (owing to BdG redundancy)

$$\gamma^\dagger_{+,\mathbf{p}\uparrow} = \gamma_{-,-\mathbf{p}\downarrow}$$

$$\gamma^\dagger_{+,\mathbf{p}\downarrow} = \gamma_{-,-\mathbf{p}\uparrow}$$

Coherence factors give the difference
Between particle and hole weights

$$\tan a_{\mathbf{p}} = \frac{\epsilon(p) + \sqrt{\epsilon(p)^2 + |\Delta|^2}}{|\Delta|}$$

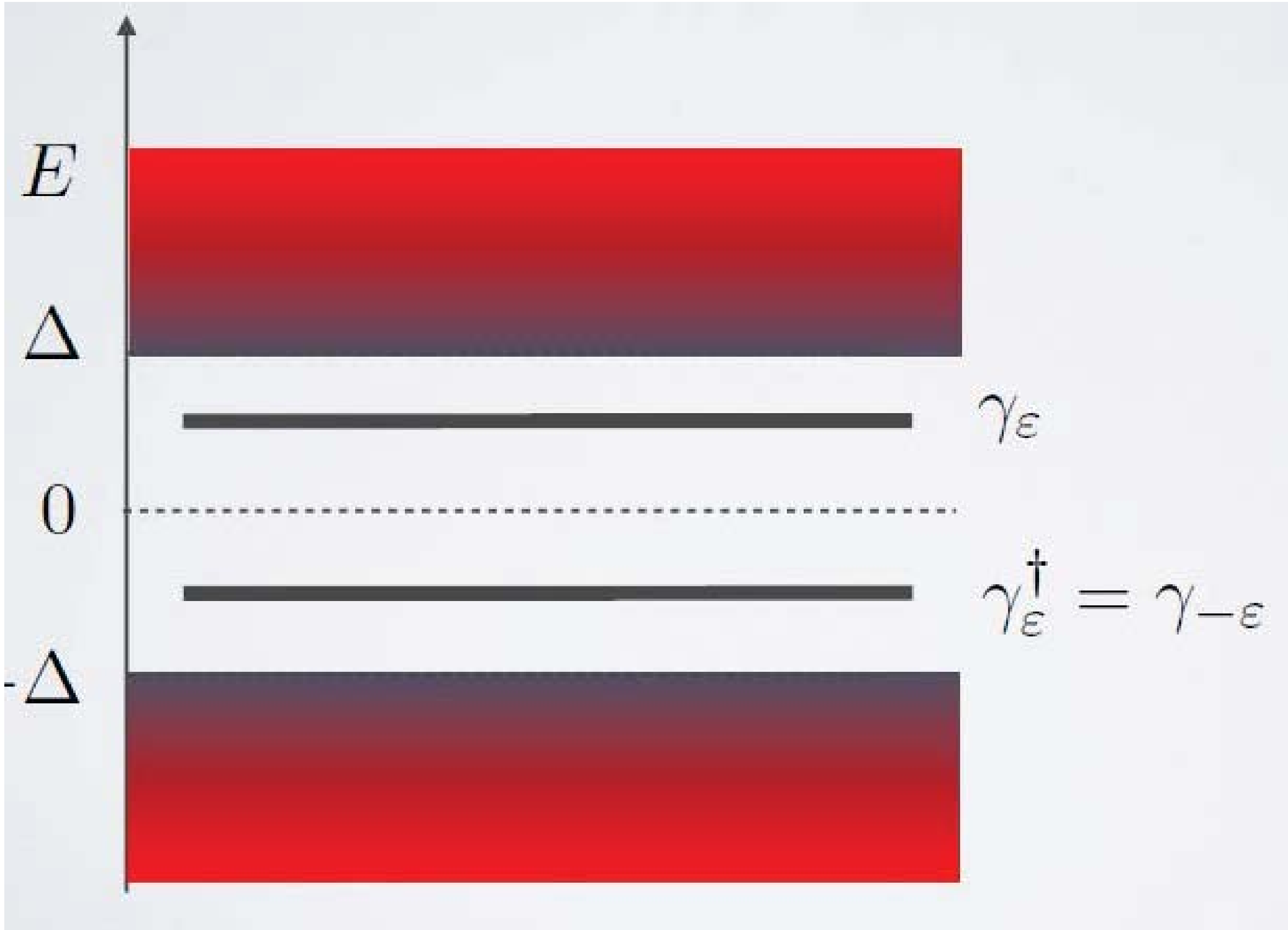
$$\tan \beta_{\mathbf{p}} = \frac{\epsilon(p) - \sqrt{\epsilon(p)^2 + |\Delta|^2}}{|\Delta|}$$

NOTE

At zero energy (mid-gap) **equal weight** superpositions of electrons and holes

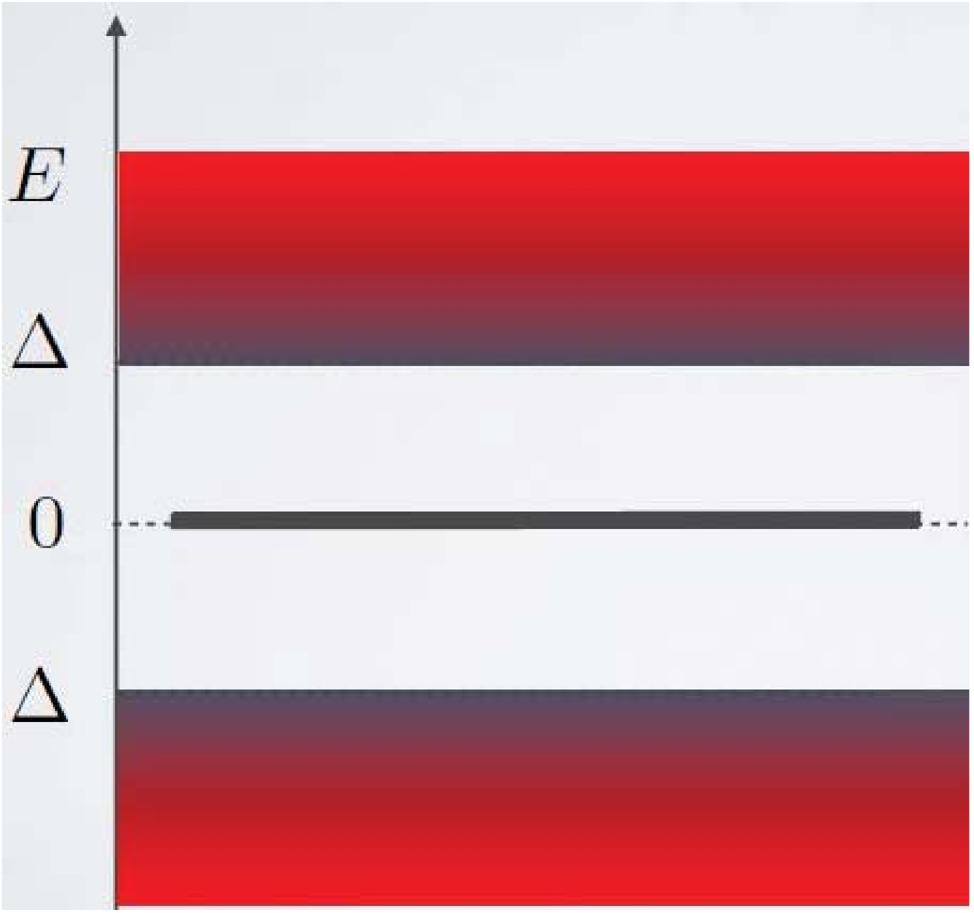
BdG Spectrum

Eigenvalues of the Bogoliubov-De Gennes equation
come in pairs due to particle-hole symmetry



<u>Non-degenerate</u> zero modes correspond to charge neutral superpositions of electrons and holes

= Majorana fermions



$$\gamma_0^\dagger = \gamma_0$$

s-wave superconductors

Symmetry of pairing: Pauli exclusion principle imposes that the pairing function must be antisymmetric.

$$\Delta_{\alpha,\beta}(k) \propto \langle c_\alpha(k) c_\beta(-k) \rangle = -\Delta_{\beta,\alpha}(-k)$$

$$\Delta_{\alpha,\beta}(k) = f_{\alpha,\beta} \Delta(k)$$

Singlet pairing: spin part odd, orbital part even

$$\Delta(k) = \Delta(-k)$$

s-wave superconductor

p-wave superconductors

Symmetry of pairing: Pauli exclusion principle imposes that the pairing function must be antisymmetric.

$$\Delta_{\alpha,\beta}(k) \propto \langle c_\alpha(k) c_\beta(-k) \rangle = -\Delta_{\beta,\alpha}(-k)$$

$$\Delta_{\alpha,\beta}(k) = f_{\alpha,\beta} \Delta(k)$$

Triplet pairing: spin part even, orbital part odd

$$\Delta(k) = -\Delta(-k)$$

p-wave superconductor

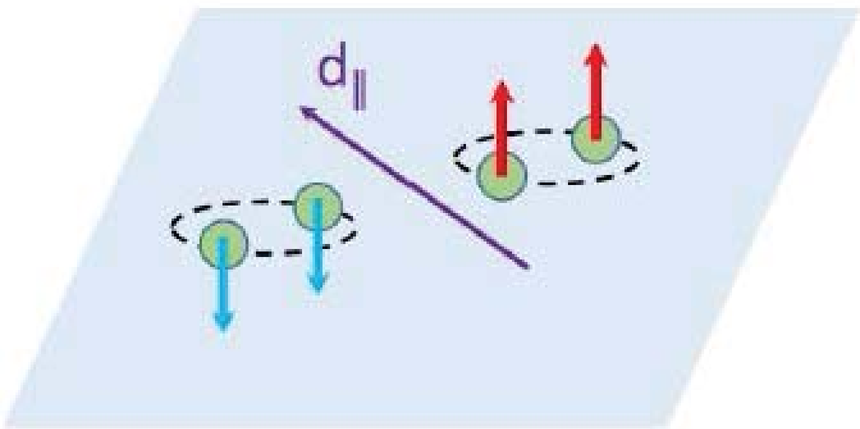
d vector representation

order parameter : $\psi = (\uparrow,\downarrow)_1 \begin{pmatrix} \Delta_{\uparrow\uparrow} & \Delta_{\uparrow\downarrow} \\ \Delta_{\uparrow\downarrow} & \Delta_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}_2$

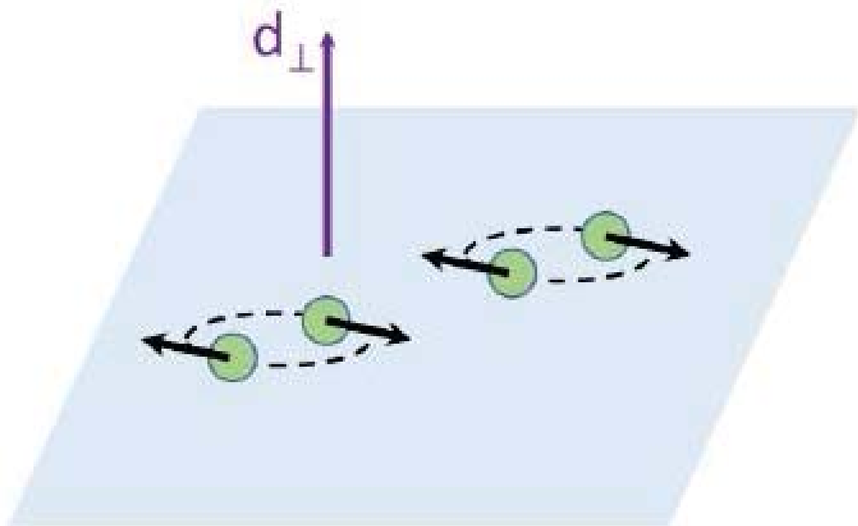
can organize into 3 - component vector $\psi = i\{\vec{d}(\vec{k})\cdot\vec{\sigma}\}\sigma_y$

Balian-Wethamer, 1963

$$\psi = i\{\vec{d}(\vec{k})\cdot\vec{\sigma}\}\sigma_y = \begin{pmatrix} -d_x + id_y & d_z \\ d_z & d_x + id_y \end{pmatrix}$$



$|\uparrow\uparrow\rangle\ ,\ |\downarrow\downarrow\rangle$



$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \Rightarrow S_z = 0$

About fundamental symmetries

The p/h operator Ξ acts on the Hamiltonian as follows:

$$\Xi H(k)\Xi^{-1} = -H(-k) \qquad \Xi^2 = +1$$

Time reversal symmetry operator Θ $\qquad \Theta^2 = \pm 1$

Time reversal symmetric Hamiltonian: $\Theta H(k)\Theta^{-1} = H(-k)$

Three different classes depending on Θ^2

Classe	Θ^2	$d=1$	$d=2$	$d=3$
BDI	$+1$	$\mathbb{Z}$	0	0
D	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

$\Theta^2=0$ means no time reversal symmetry

Ia) 1D spinless topological superconductors

Simplest model: 1D spinless SC

$$H = \sum_p c_p^\dagger \left(\frac{p^2}{2m} - \mu \right) c_p.$$

$$H_\Delta = \frac{1}{2} \left(\Delta p c_p^\dagger c_{-p}^\dagger + \Delta^* p c_{-p} c_p \right)$$

$$H_{\mathrm{BdG}} = \sum_p \frac{1}{2} \Psi_p^\dagger \begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta p \\ \Delta^* p & -\frac{p^2}{2m} + \mu \end{pmatrix} \Psi_p; \qquad \Psi_p = (c_p \;\; c_{-p}^\dagger)^T$$

$$E_\pm = \pm \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + |\Delta|^2 p^2}$$

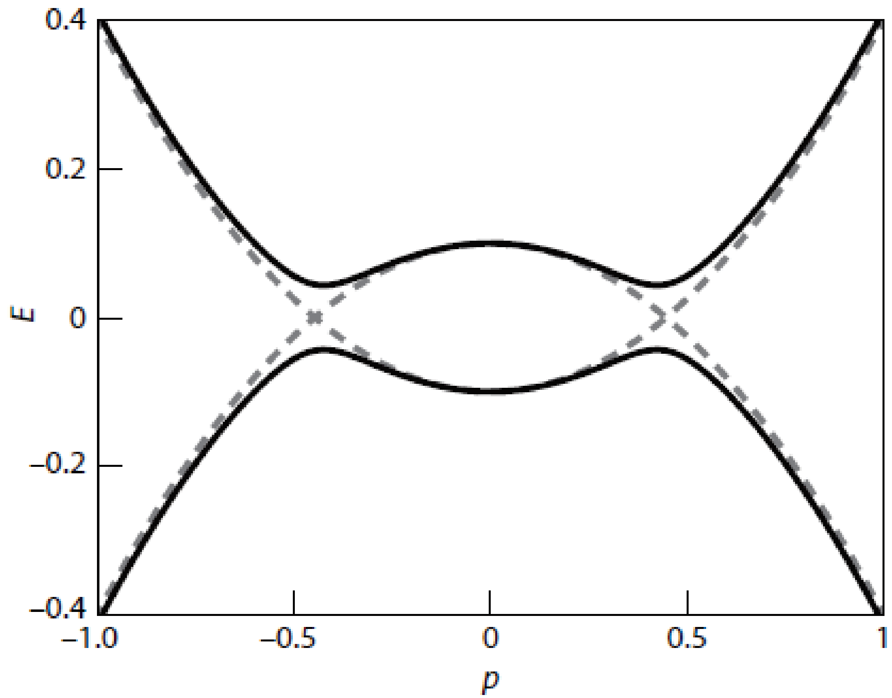
For μ different from 0, the system is always gapped

Simplest model: 1D spinless SC

$$E_{\pm} = \pm\sqrt{(\frac{p^2}{2m} - \mu)^2 + |\Delta|^2 p^2}$$

μ < 0 : the system without pairing is insulating → Strong pairing phase

μ > 0 : the system without pairing is metallic → Weak pairing phase



The Kitaev chain: a 1D p-wave superconductor

$$H = -\sum_{i=1}^{N-1}\left[t c_i^\dagger c_{i+1} + \Delta c_i^\dagger c_{i+1}^\dagger + \text{h.c.}\right] - \mu \sum_{i=1}^{N} n_i$$

$$H_{\text{BdG}} = \frac{1}{2}\sum_p \Psi_p^\dagger \begin{pmatrix} -2t\cos p - \mu & 2i|\Delta|\sin p \\ -2i|\Delta|\sin p & 2t\cos p + \mu \end{pmatrix} \Psi_p$$

$$E_\pm(p) = \pm\sqrt{(2t\cos p + \mu)^2 + 4|\Delta|^2 \sin^2 p}$$

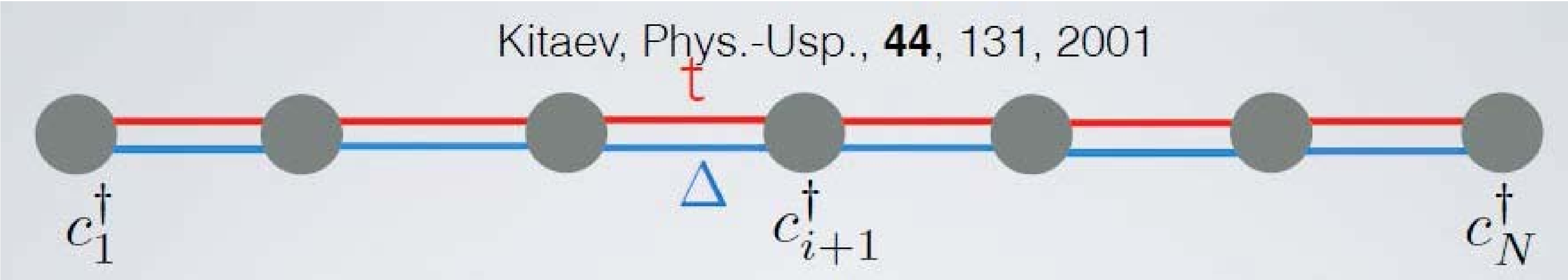
The gap closes for $|\mu| = 2t$

$|\mu| > 2t$: the system without pairing is insulating → Strong (Trivial) pairing phase

$|\mu| < 2t$: the system without pairing is metallic → Weak (topological) pairing phase

Let us analyze the real space Hamiltonian

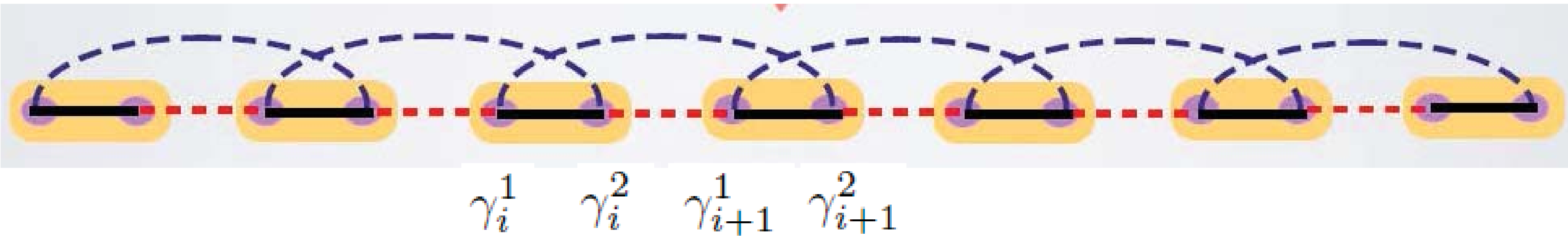
Kitaev's model



Majorana decomposition

$$\gamma_i^1 = c_i + c_i^\dagger$$

$$\gamma_i^2 = i(c_i - c_i^\dagger)$$



Any fermionic Hamiltonian can be recast in terms of Majorana operators !

Kitaev's model

However, very few can support solutions with isolated localized Majorana fermions

The necessary magic trick for getting a majorana fermion

Kitaev's model

$\gamma_i^1 \quad \gamma_i^2 \quad \gamma_{i+1}^1 \quad \gamma_{i+1}^2$

$$H = -\sum_{i=1}^{N-1}\left[t c_i^\dagger c_{i+1} + \Delta c_i^\dagger c_{i+1}^\dagger + \text{h.c.}\right] - \mu \sum_{i=1}^{N} n_i$$

$$H = -i\sum_{i=1}^{N-1}[\omega_+ \gamma_i^2 \gamma_{i+1}^1 - \omega_- \gamma_i^1 \gamma_{i+1}^2] - i\frac{\mu}{2}\sum_{i=1}^{N} \gamma_i^2 \gamma_i^1$$

$$\omega_+ = \frac{1}{2}(t+\Delta)$$

$$\omega_- = \frac{1}{2}(t-\Delta)$$

$$\mu$$

Kitaev's model

$$t = \Delta = 0$$

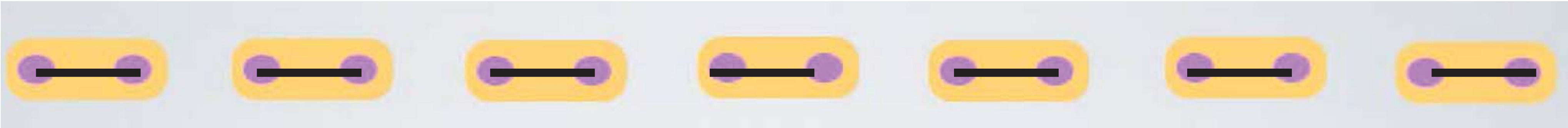
$$H = -i \sum_{i=1}^{N-1} [\cancel{\omega_+ \gamma_i^2 \gamma_{i+1}^1} - \cancel{\omega_- \gamma_i^1 \gamma_{i+1}^2}] - i\frac{\mu}{2} \sum_{i=1}^{N} \gamma_i^2 \gamma_i^1$$

$$\omega_+ = t = \Delta = 0$$

$$\omega_- = 0$$

Kitaev's model

t= Δ=0



$\gamma_i^1 \quad \gamma_i^2 \quad \gamma_{i+1}^1 \quad \gamma_{i+1}^2$

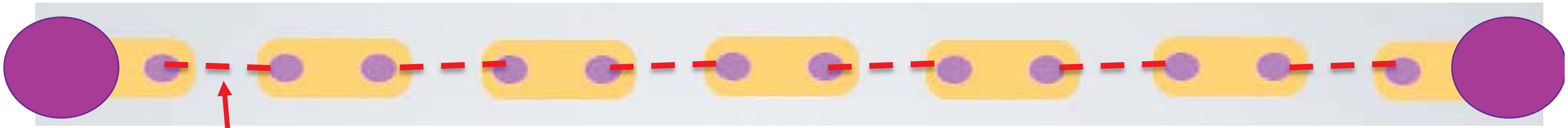
$$H = -i\frac{\mu}{2}\sum_{i=1}^{N}\gamma_i^2\gamma_i^1 = -\mu\sum_{i=1}^{N}(c_i^\dagger c_i - \frac{1}{2})$$

TRIVIAL NONINTERACTING FERMIONS ON THE LATTICE

$|\mu| > 2t$

Kitaev's model

$t = \Delta \;\; ; \quad \mu = 0 \qquad |\mu| < 2t$



$\gamma_i^1 \quad \gamma_i^2 \quad \gamma_{i+1}^1 \quad \gamma_{i+1}^2$

fuse Majorana fermions across nearest neighbor bonds

$$a_i = \frac{1}{2}(\gamma_i^2 + i\gamma_{i+1}^1)$$

$$a_i^\dagger = \frac{1}{2}(\gamma_i^2 - i\gamma_{i+1}^1)$$

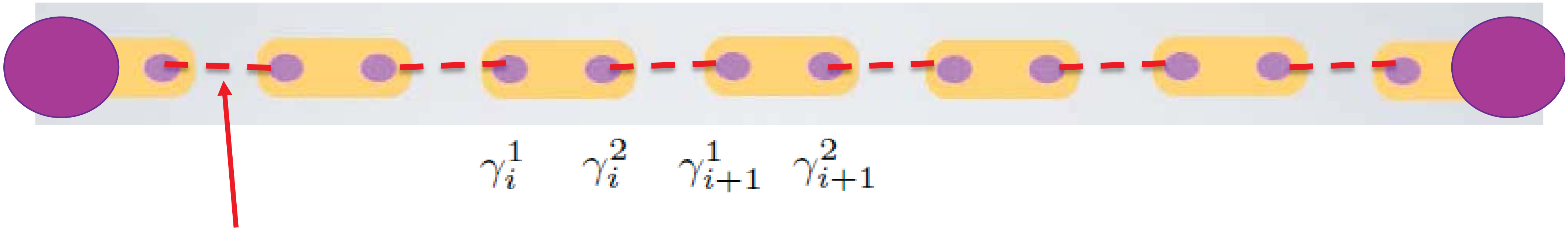
$$\Rightarrow \quad H = 2t \sum_{i=1}^{N-1} (a_i^\dagger a_i - \frac{1}{2})$$

$$H = -i \sum_{i=1}^{N-1} [\omega_+ \gamma_i^2 \gamma_{i+1}^1 \cancel{- \omega_- \gamma_i^1 \gamma_{i+1}^2}] \cancel{- i\frac{\mu}{2} \sum_{i=1}^{N} \gamma_i^2 \gamma_i^1}$$

$$\omega_+ = t = \Delta$$

Kitaev's model

t= Δ ; μ=0 |μ| < 2t



fuse Majorana fermions across nearest neighbor bonds

$$a_i = \frac{1}{2}(\gamma_i^2 + i\gamma_{i+1}^1)$$

$$a_i^\dagger = \frac{1}{2}(\gamma_i^2 - i\gamma_{i+1}^1)$$

$$H = 2t\sum_{i=1}^{N-1}(a_i^\dagger a_i - \frac{1}{2})$$

GAPPED SPECTRUM+ZERO-ENERGY MAJORANAS AT THE END OF THE WIRE
(DECOUPLED FROM THE BULK OF THE CHAIN)!!!

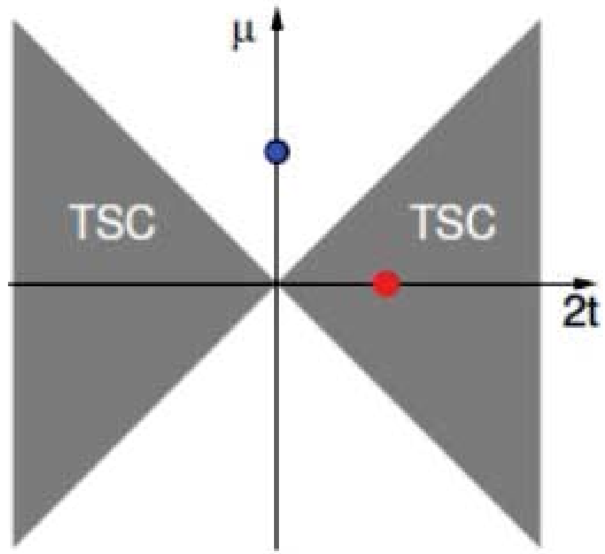
Phase diagram of the 1D Kitaev model

$$H = -\sum_{i=1}^{N-1}\left[tc_i^\dagger c_{i+1} + \Delta c_i^\dagger c_{i+1}^\dagger + \text{h.c.}\right] - \mu\sum_{i=1}^{N} n_i$$

+ periodic boundary conditions

$$\Rightarrow \quad H_{\text{BdG}} = \frac{1}{2}\sum_p \Psi_p^\dagger \begin{pmatrix} -2t\cos p - \mu & 2i|\Delta|\sin p \\ -2i|\Delta|\sin p & 2t\cos p + \mu \end{pmatrix} \Psi_p$$

The gap closes for |μ|= 2t



System is topologically non trivial (topological SC) for |μ| < 2t

Z_2 Bulk invariant

$$H = -i\sum_{i=1}^{N-1}[\omega_+\gamma_i^2\gamma_{i+1}^1 - \omega_-\gamma_i^1\gamma_{i+1}^2] - i\frac{\mu}{2}\sum_{i=1}^{N}\gamma_i^2\gamma_i^1$$

MOMENTUM SPACE

$$H = \frac{i}{4}\sum_q \begin{pmatrix}\gamma_q^1 & \gamma_q^2\end{pmatrix}\begin{pmatrix}0 & D_q \\ -D_q^* & 0\end{pmatrix}\begin{pmatrix}\gamma_{-q}^1 & \gamma_{-q}^2\end{pmatrix}$$

with $\quad D_q = -\mu - 2t\cos(q) - 2i\Delta\sin(q)$

$$M = sgn\{D_0 D_\pi\} = sgn(\mu^2 - 4t^2)$$

Z_2 Bulk invariant

The topological number characterizing the systems is a $\mathbb{Z}_2$ invariant

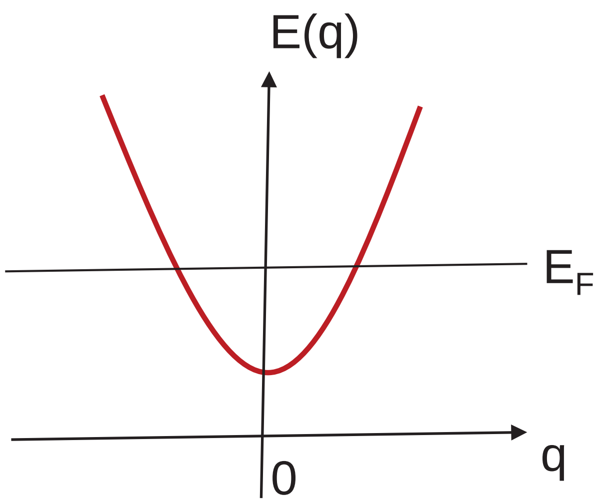
$$\mathcal{M}(H) = \mathrm{sgn}(\mathrm{Pf}\,\tilde{B}(0))\,\mathrm{sgn}(\mathrm{Pf}\,\tilde{B}(\pi))$$

where B matrix representation of H in the Majorana basis

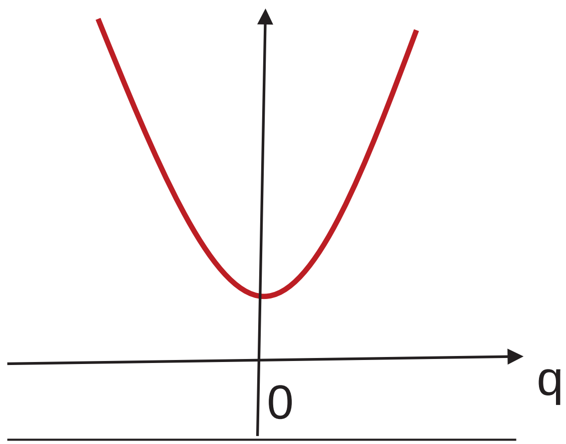
Actually when superconducting is weak, we can calculate $\mathcal{M}(H)$ in absence of the pairing term

$$\mathcal{M}(H) = \mathcal{M}(H_0) = (-1)^{\nu(\pi)-\nu(0)}$$

Number of Fermi points in the interval [0 , π]



TOPOLOGICAL



TRIVIAL

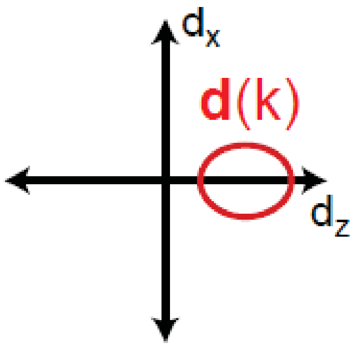
Z Bulk invariant

$$H-\mu N=\sum_i t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)-\mu c_i^\dagger c_i+\Delta(c_i c_{i+1}+c_{i+1}^\dagger c_i^\dagger)$$

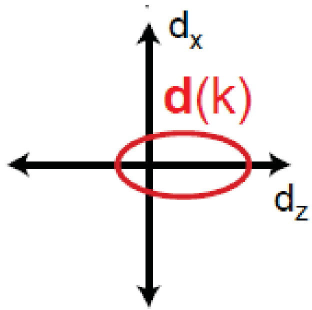
$$H_{BdG}(k)=\tau_z(2t\cos k-\mu)+\tau_x\Delta\sin k=\mathbf{d}(k)\cdot\vec{\tau}$$

Only two components of **d** appear !

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor

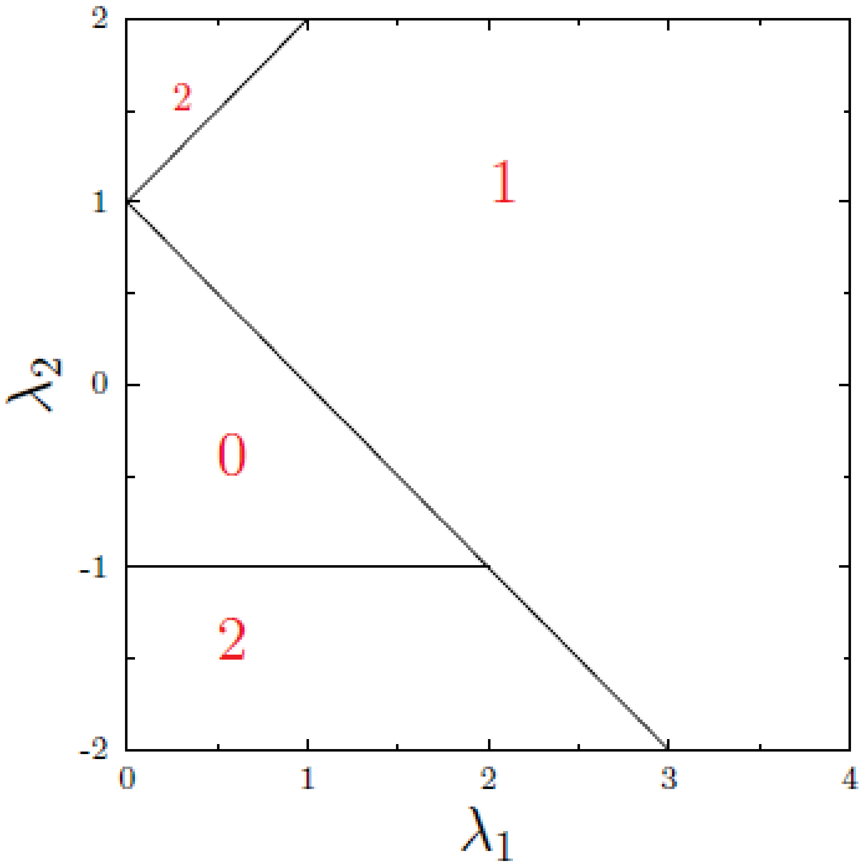


Winding number = Z invariant (emergent chiral symmetry)

BDI class

Extended Kitaev model

$$H=\sum_j\big[-\mu(1-2c_j^\dagger c_j)-\lambda_1(c_j^\dagger c_{j+1}+c_j^\dagger c_{j+1}^\dagger+\mathrm{H.c.})$$
$$-\lambda_2(c_{j-1}^\dagger c_{j+1}+c_{j-1}^\dagger c_{j+1}^\dagger+\mathrm{H.c})\big]$$

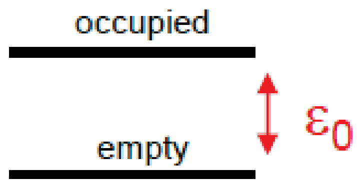


K. Niu et al., Phys Rev. B 85, 035110 (2012)

About Majorana fermions'properties

Two Majorana fermions define a single two level system

$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger \\ \gamma_2 = -i(\Psi - \Psi^\dagger) \end{cases} \Rightarrow H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$



- 2 degenerate states (full/empty) = 1 qubit

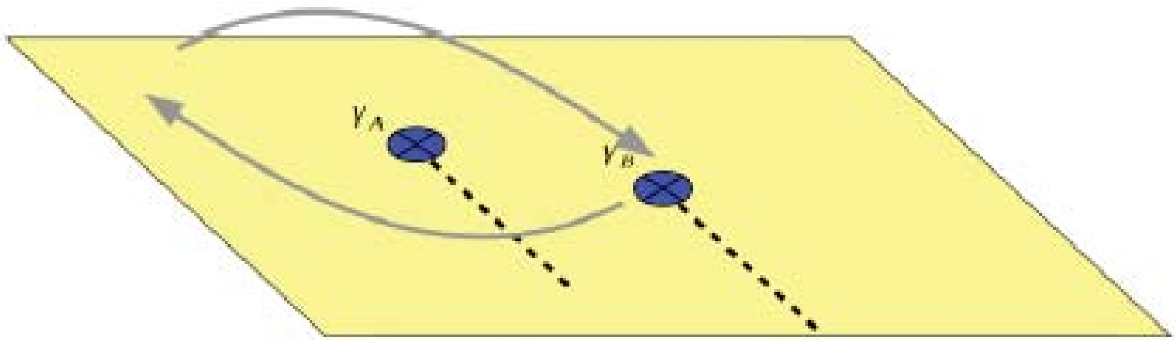
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally : Immune from local decoherence

Braiding performs unitary operations:
Non-Abelian statistics :

Interchange rule

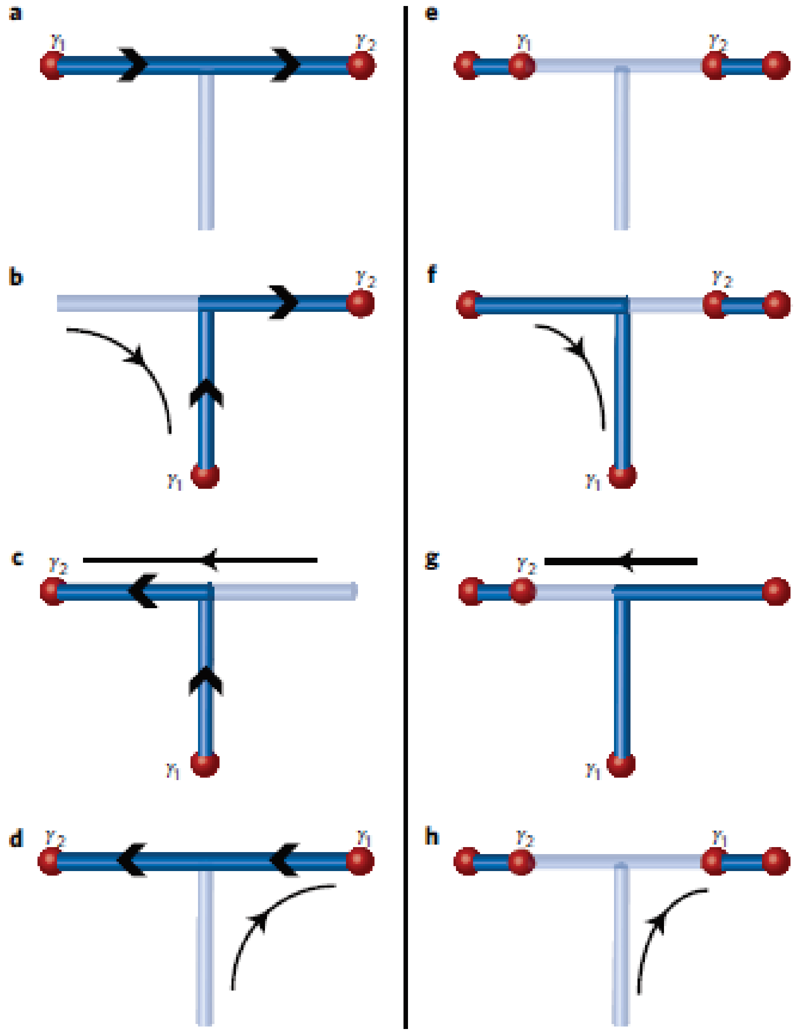
$$\gamma_i \rightarrow \gamma_j$$

$$\gamma_j \rightarrow -\gamma_i$$



- exchange (=π rotation): $\gamma_b \rightarrow \gamma_a \qquad \gamma_a \rightarrow -\gamma_b$
- braid around (=2π rotation): $\gamma_a \rightarrow -\gamma_a \qquad \gamma_b \rightarrow -\gamma_b$

Braiding of Majorana fermions



T-junctions shows non-Abelian statistics

Alicea et al., Nature Physics, 2010

Ib) Physical realizations of 1D topological superconductors

Challenges

- Electrons are spin-degenerate so we must freeze out half of the degrees of freedom to have an effective spinless system.

- p-wave superconductors seem rather rare in nature

→ Clever proposals that overcome these challenges have the same three main ingredients:

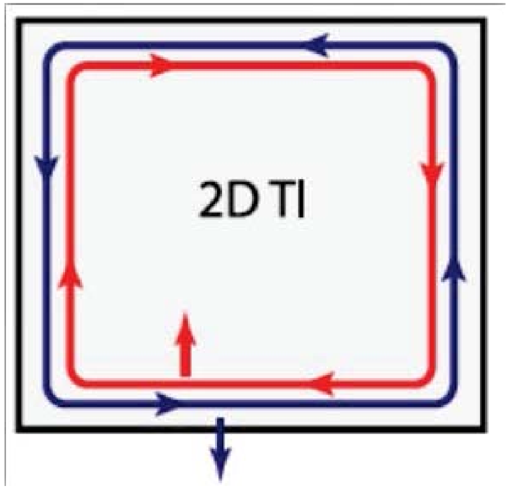
1. Instead of using intrinsic superconductivity use the superconducting proximity effect.

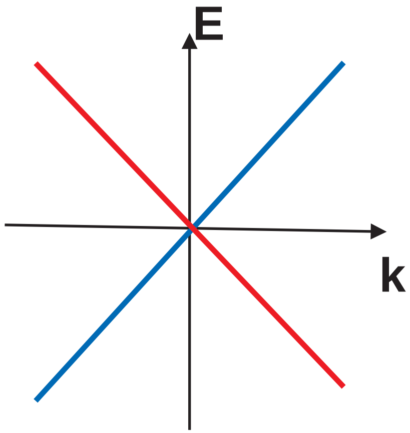
2. Time-reversal symmetry breaking

3. Spin-orbit coupling or magnetic texture

USE TOPOLOGICAL INSULATORS !

From 2D topological insulators





$$H_{2D\ TI} = \int dx\, \psi^\dagger(-iv\sigma^z\partial_x - \mu)\psi$$

Add a Zeeman term: $H_Z = -h\int dx\, \psi^\dagger\sigma^x\psi$

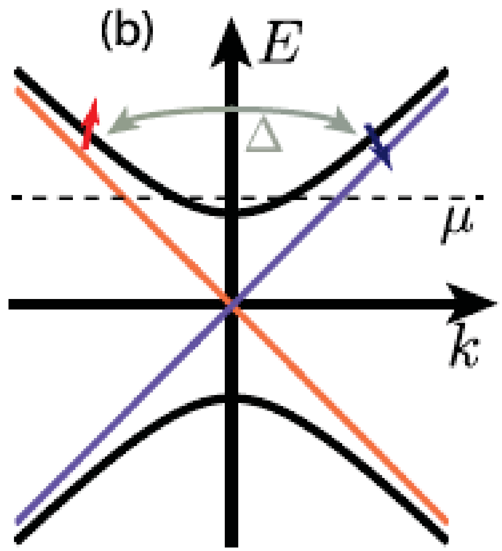
A gap opens owing to a mass term: Zeeman interaction that breaks time-reversal symmetry.

Edge spectrum becomes : $\epsilon_\pm(k) = -\mu \pm \sqrt{(vk)^2 + h^2}$

Add pairing by proximity effect of a s-wave superconductor

$$H_\Delta = \Delta\int dx\, (\psi_\uparrow^\dagger\psi_\downarrow^\dagger + h.c.)$$

From 2D topological insulators



The total Hamiltonian in the new basis acquires p-wave pairing terms !!!

$$H' = \int \frac{dk}{2\pi} \{ \epsilon_+(k)\psi_+^\dagger(k)\psi_+(k) + \epsilon_-(k)\psi_-^\dagger(k)\psi_-(k)$$

$$+ \frac{\Delta_p(k)}{2}[\psi_+(-k)\psi_+(k) + \psi_-(-k)\psi_-(k) + H.c.]$$

$$+ \Delta_s(k)[\psi_-(-k)\psi_+(k) + H.c.]\}$$

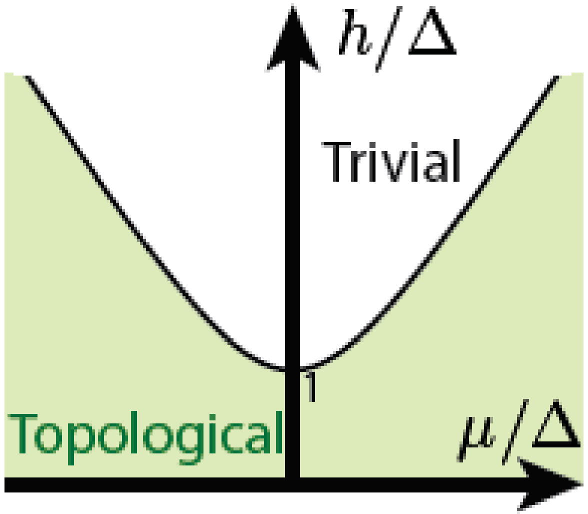
$$\Delta_p(k) = \frac{vk\Delta}{\sqrt{(vk)^2 + h^2}}; \quad \Delta_s(k) = \frac{h\Delta}{\sqrt{(vk)^2 + h^2}}$$

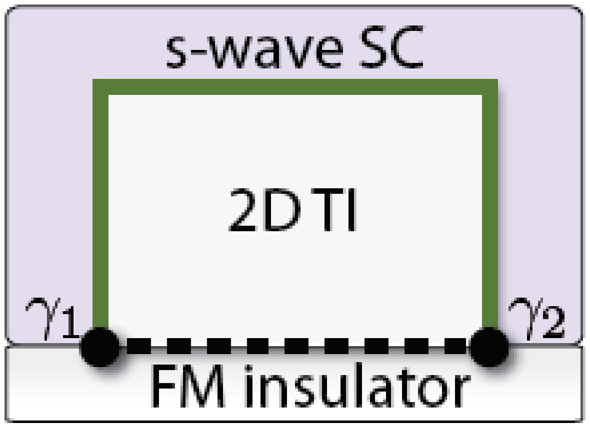
$$\Rightarrow \quad E'_\pm(k) = \sqrt{\Delta^2 + \frac{\epsilon_+^2 + \epsilon_-^2}{2} \pm (\epsilon_+ - \epsilon_-)\sqrt{\Delta_s^2 + \mu^2}}$$

From 2D topological insulators

$$E'_{\pm}(k) = \sqrt{\Delta^2 + \frac{\epsilon_+^2 + \epsilon_-^2}{2} \pm (\epsilon_+ - \epsilon_-)\sqrt{\Delta_s^2 + \mu^2}}$$

Gap closing: $h^2 = \Delta^2 + \mu^2$





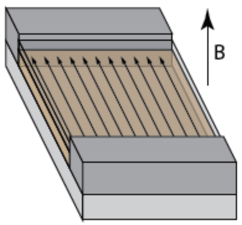
Fu and Kane, PRL100, 096407 2008

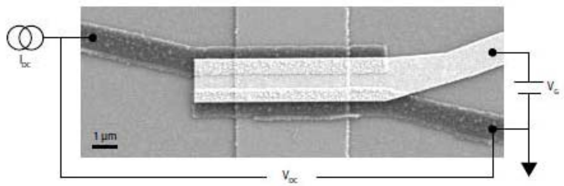
From Alicea, Rep.Prog. Phys. 75, 076501 (2012)

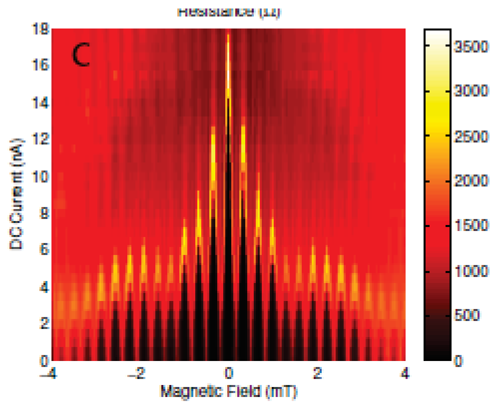
Proposed realizations for a 1D topological SC

- Topological insulators in proximity of superconductors

➡ Possible experimental signatures of proximized SC







Induced SC in the anomalous spin Hall effect: observation of Josephson supercurrent in the helical edges channels via the Fraunhofer interference pattern

Molenkamp & Yacoby groups, Nature Physics 2014

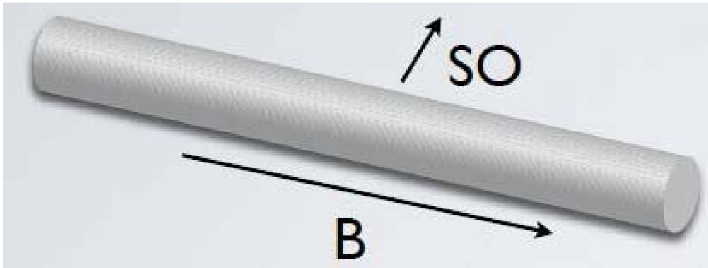
See Tristan's lectures

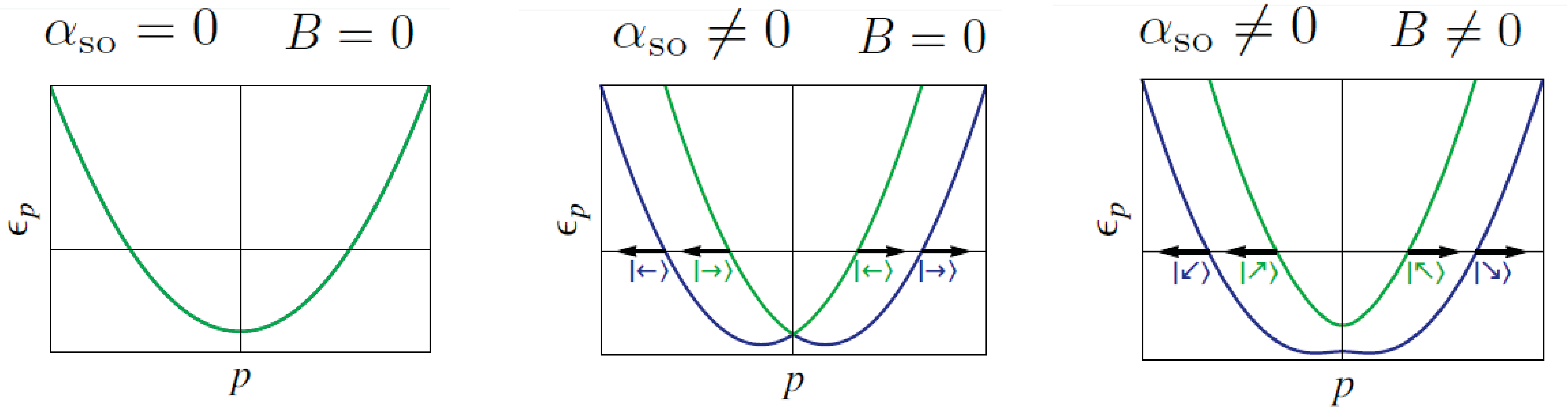
Proposed realizations for a 1D topological SC

1D quantum semiconducting nanowires
In proximity of a s-wave superconductor

Lutchyn et al., PRL 104, (2010)

Oreg, Refael, von Oppen, PRL 105, (2010)



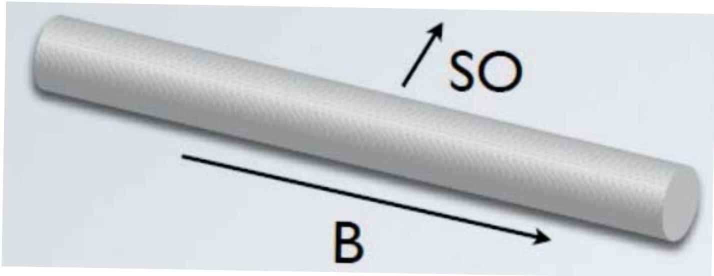


- SO splits NW states into 2 subbands of opposite helicity: + and -

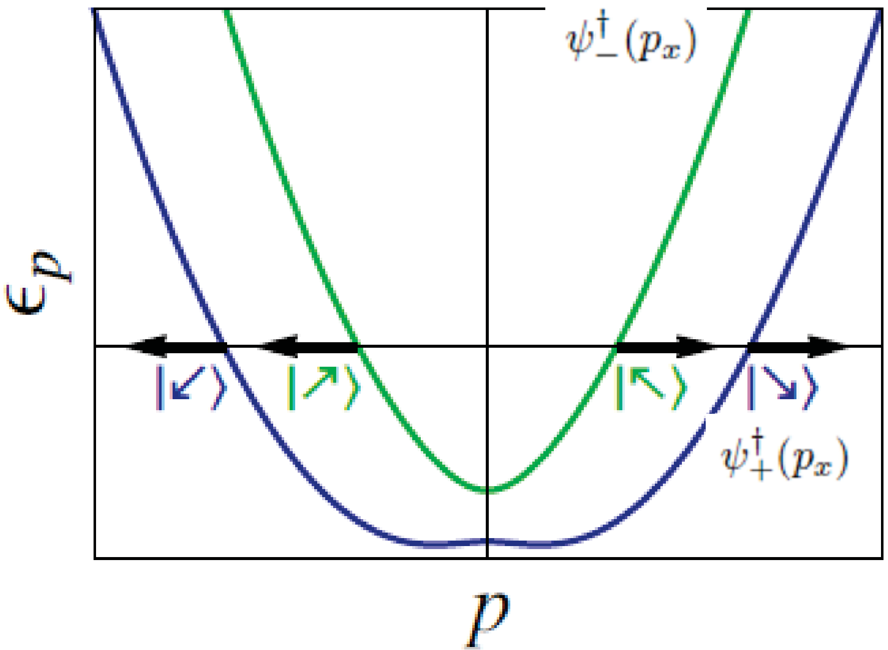
  —— +   —— -

- At finite B, +/- subbands have spins canted away from SO axis

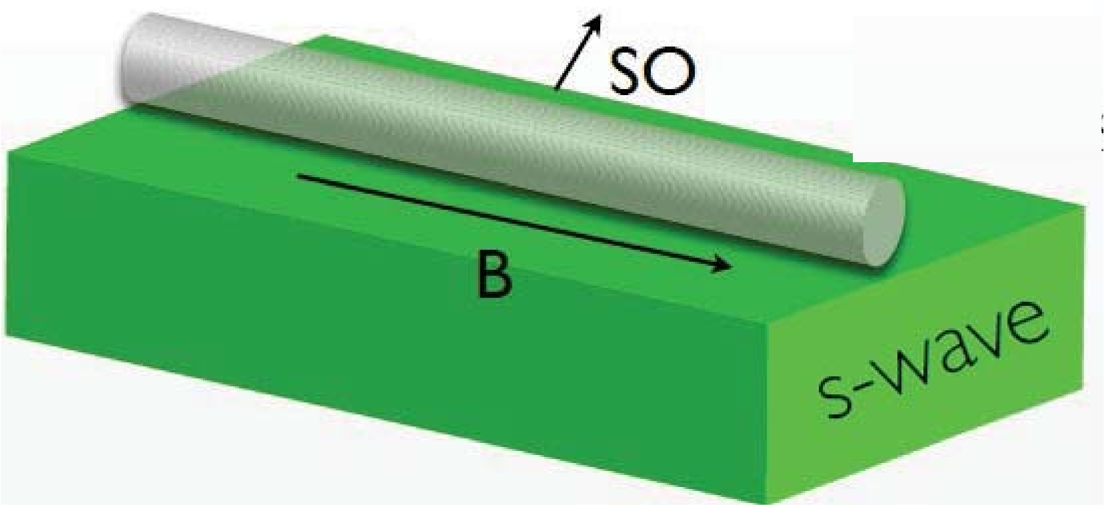
$$H_0 = \int dx\, \psi^\dagger_\sigma(x)[-\frac{\partial_x^2}{2m} - \mu + i\alpha\sigma^y\partial_x + B\sigma^x]_{\sigma\sigma'}\psi_{\sigma'}(x)$$



$\alpha_{\mathrm{SO}} \neq 0 \qquad B \neq 0$



$$H_0 = \int dx\, \psi^\dagger_\sigma(x)[-\frac{\partial_x^2}{2m} - \mu + i\alpha\sigma^y\partial_x + B\sigma^x]_{\sigma\sigma'}\psi_{\sigma'}(x)$$



$$H_\Delta = \Delta \int dx\, (\psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x) + h.c.)$$

Helical basis

$$\cancel{\Delta_{--}\psi^\dagger_-(p_x)\psi^\dagger_-(-p_x)}$$

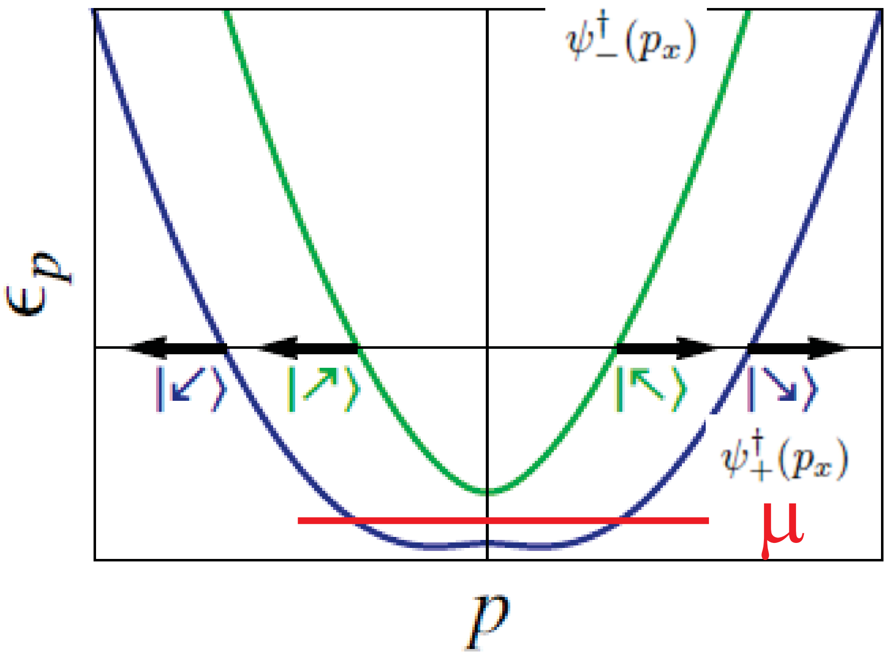
$$\cancel{\Delta_{+-}\psi^\dagger_+(p_x)\psi^\dagger_-(-p_x)}$$

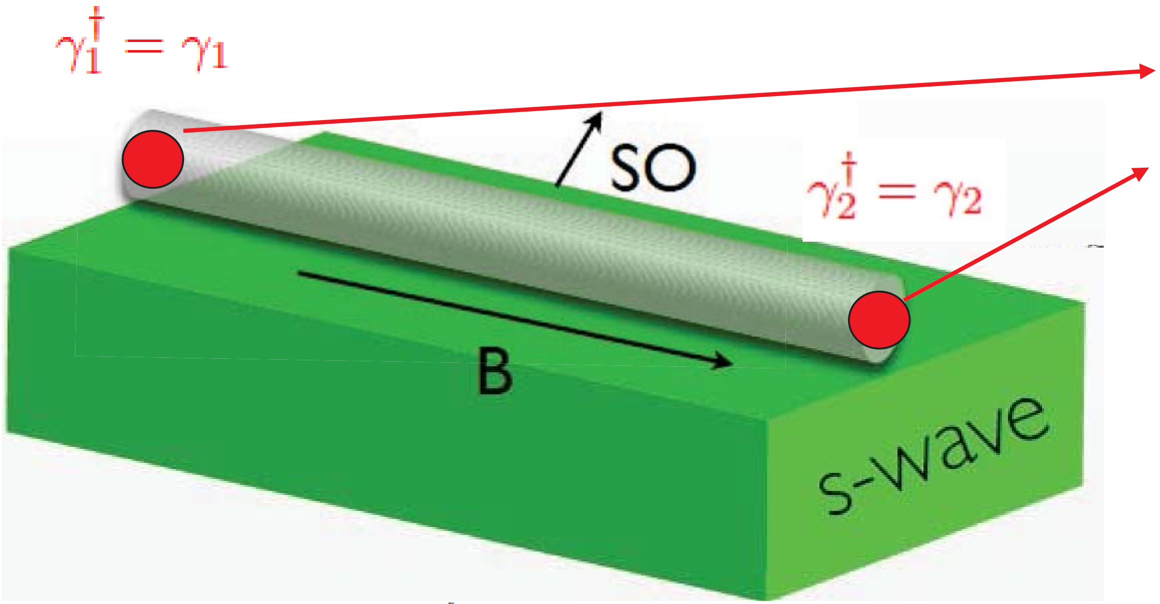
$$\Delta_{++}\psi^\dagger_+(p_x)\psi^\dagger_+(-p_x)$$

Effective p-wave pairing

$$\Delta_{++} = \frac{i\alpha p_x \Delta}{\sqrt{\alpha^2 p_x^2 + B^2}}$$

$\alpha_{\text{SO}} \neq 0 \qquad B \neq 0$



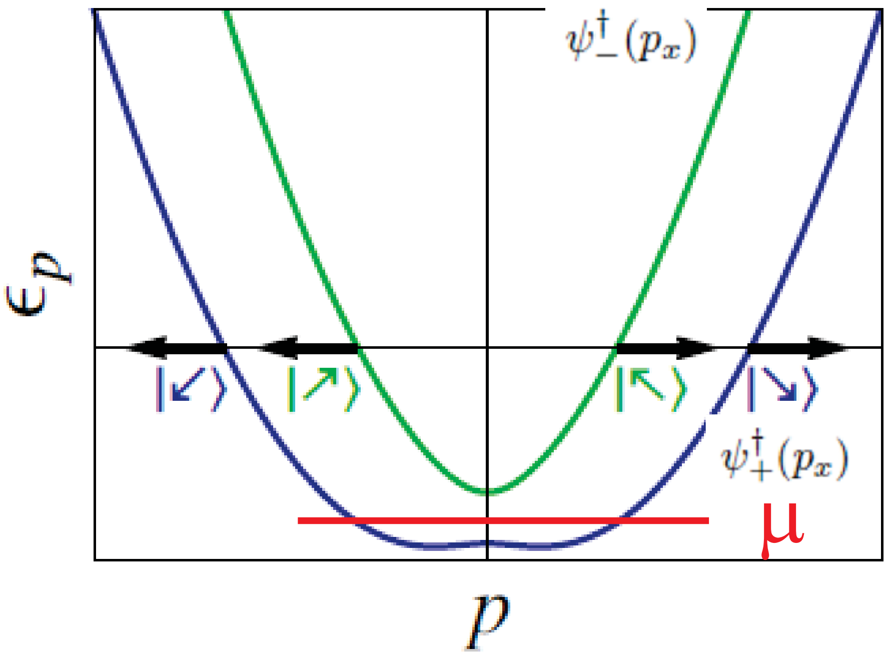


The two Majoranas constitute a *single* electronic excitation which can be *arbitrarily* delocalized.

$$i\gamma_1\gamma_2 = 2(d^\dagger d - \tfrac{1}{2})$$

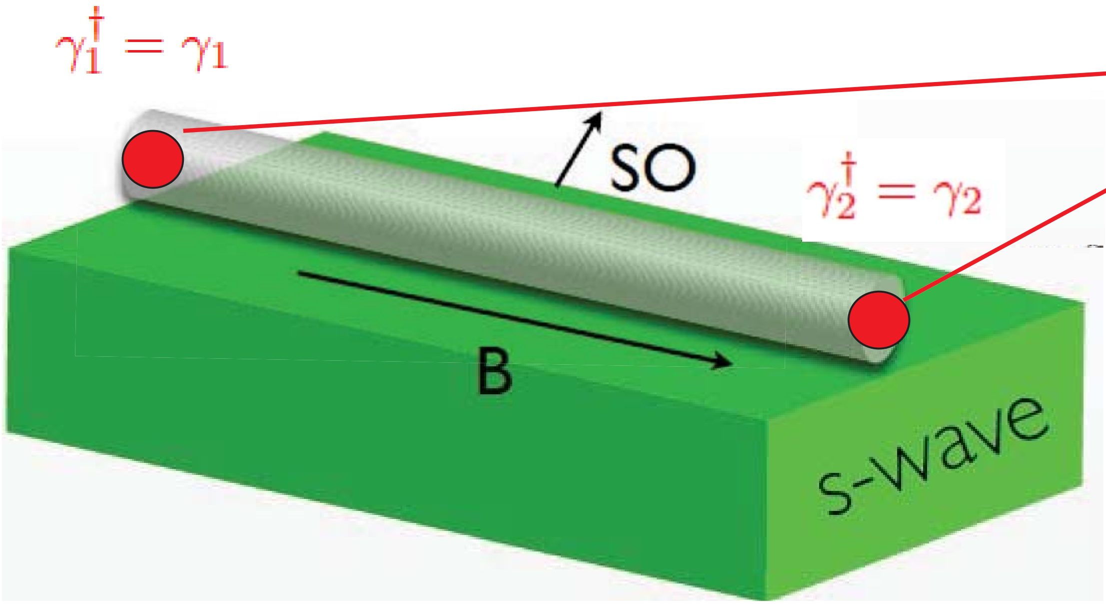
$$\alpha_{\mathrm{SO}} \neq 0 \qquad B \neq 0$$

In the presence of s-wave pairing such helical nanowire is a another realization of Kitaev's 1D p-wave superconductor model.



Effective p-wave pairing

$$\Delta_{++} = \frac{i\alpha p_x \Delta}{\sqrt{\alpha^2 p_x^2 + B^2}}$$

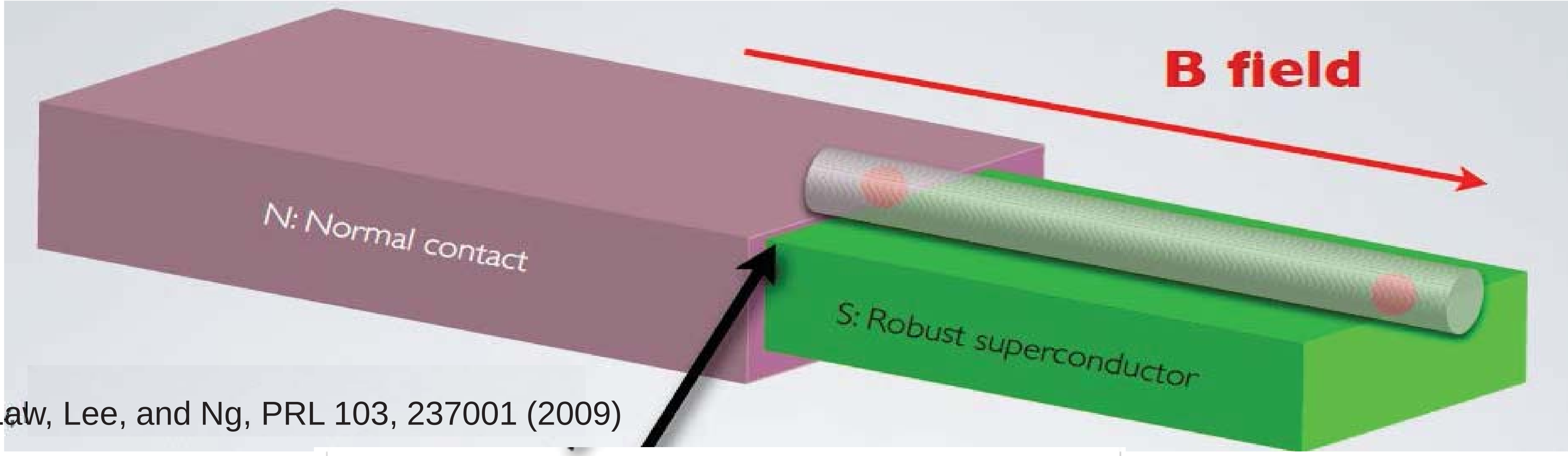


The two Majoranas constitute a *single* electronic excitation which can be *arbitrarily* delocalized.

Very attractive proposal, all the ingredients are available in the lab:

- Nanowires with strong spin-orbit coupling (In As, InSb) $\alpha_{SO} \sim 0.1 - 0.2\ eV\,\text{Å}$
- Large g-factors (g from 10 to 50)
- Good proximity effect with superconductors (Aluminium, Niobium, Vanadium, etc) with large critical fields
- Last generation: perfect epitaxial interface between InAs& Al
- Gate-tunable (low chemical potential) nanowires.

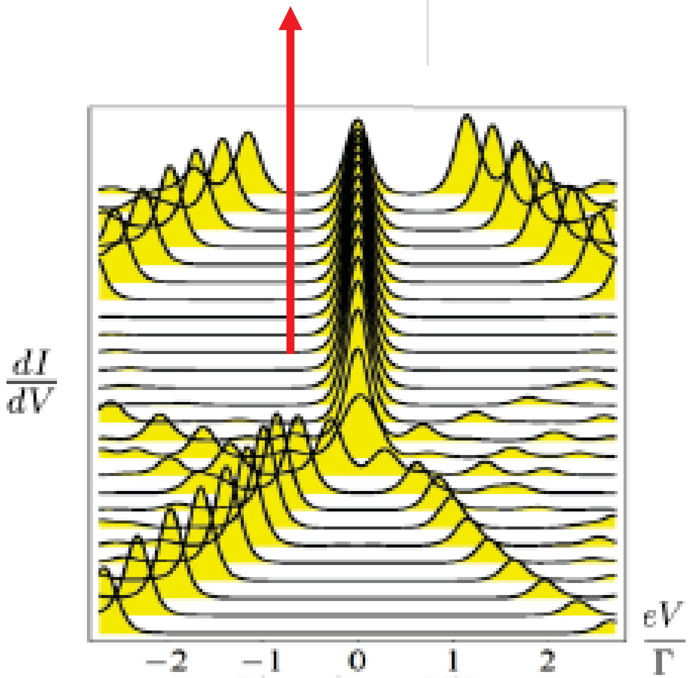
Experimental realizations



Law, Lee, and Ng, PRL 103, 237001 (2009)

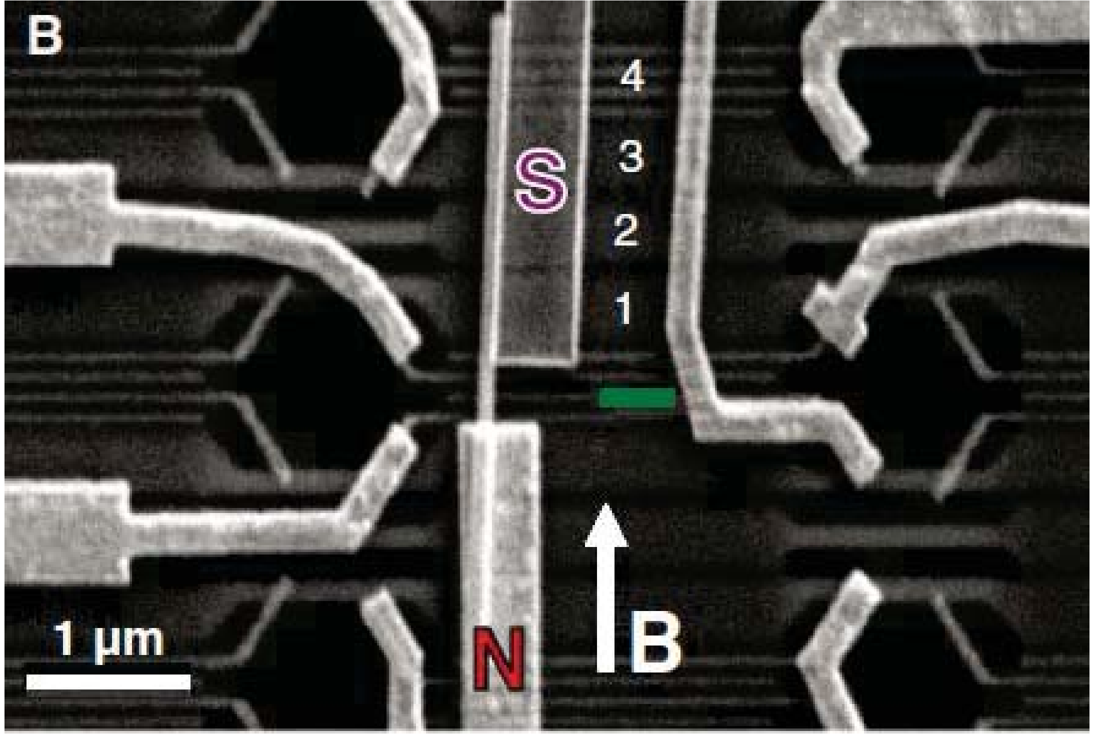
Resonant Andreev reflection in the NS interface:
owing to the presence of the Majorana bound state there is a peak at V=0. Particle-hole symmetry implies unitary transport

$$G = \frac{2e^2}{h}$$

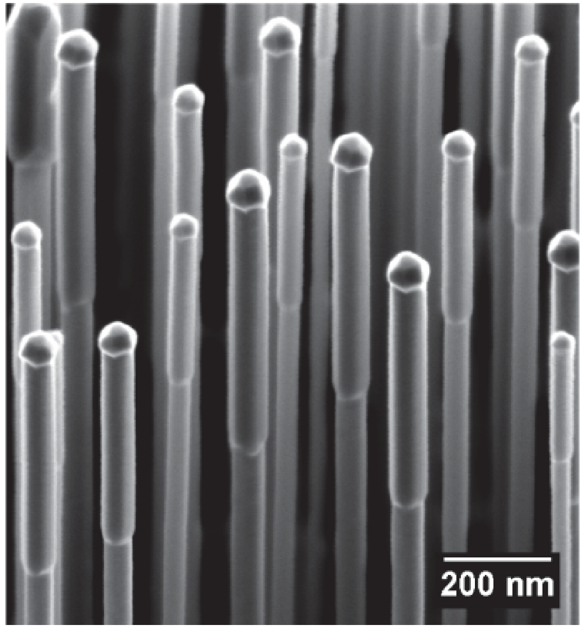


Recipe for Majoranas

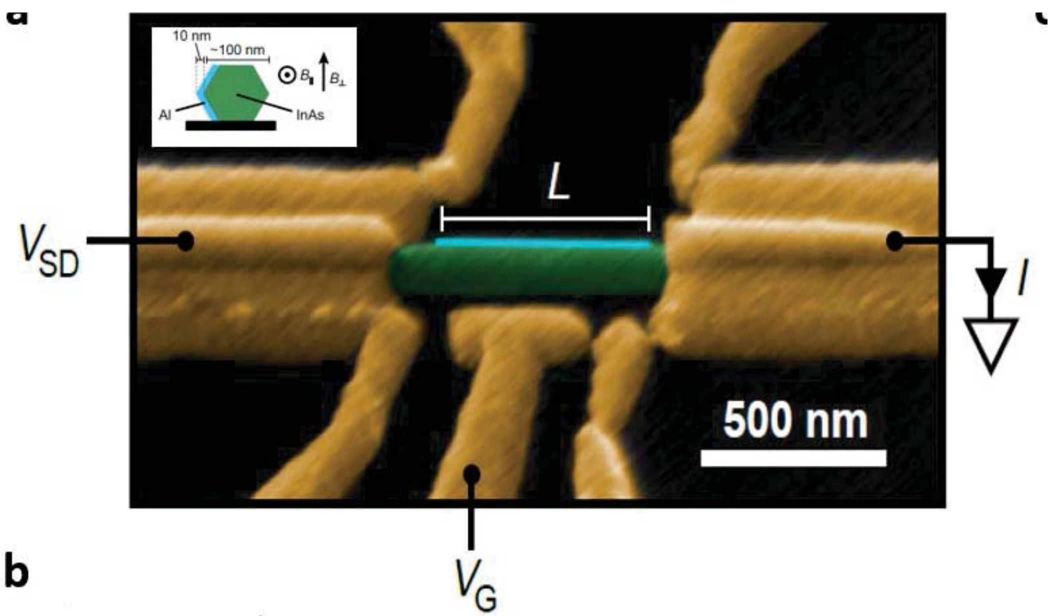
- Make a Normal-Superconductor junction with In As or InSb nanowires coupled to, say, Niobium contacts.
- Gate the system to reach low density.
- Increase magnetic field along the wire direction.
- Look for zero bias peaks in conductance that emerge after a gap closing (transition from trivial to nontrivial phase) as B field increases.



Delft first experiment (Kouwenhoven)

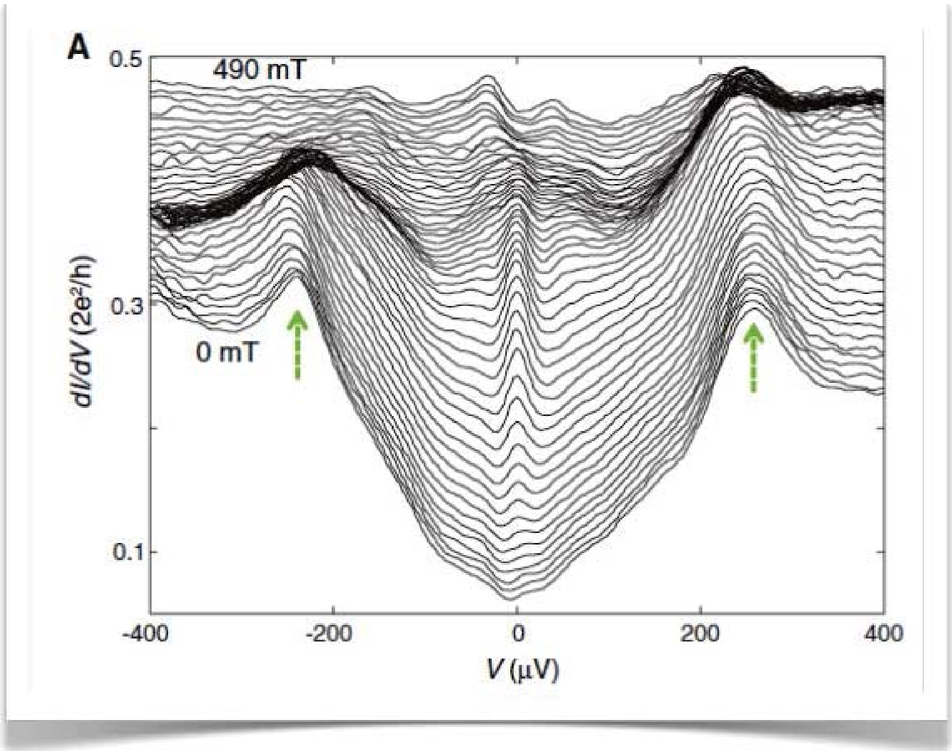


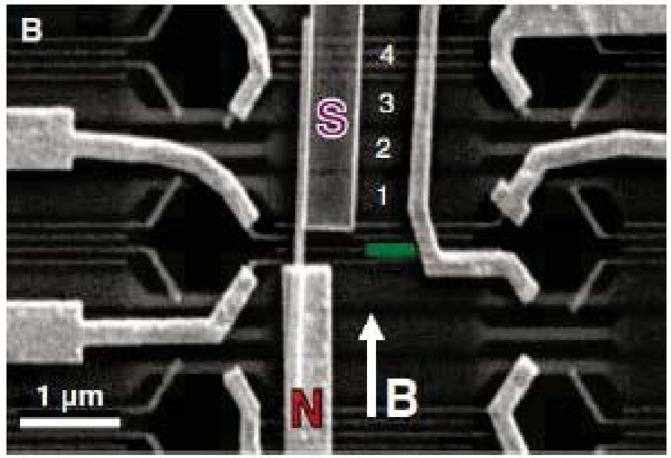
InSb nanowires
Bakkers Eindhoven-Delft)



Copenhagen experiment
(Marcus)

Zero-bias anomaly experiments : DELFT





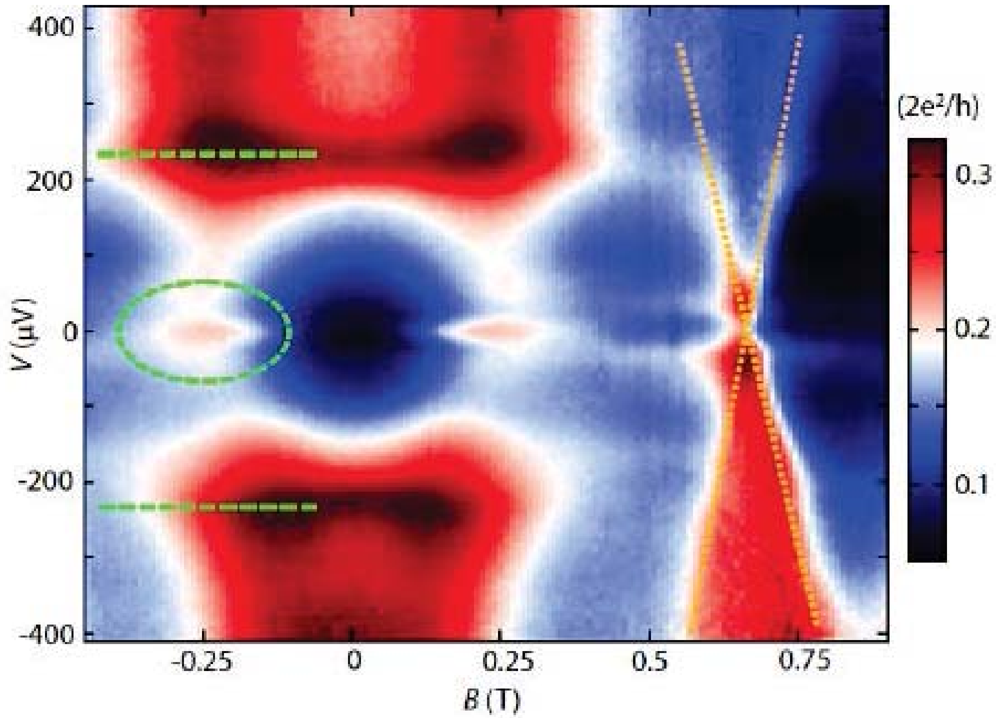
Signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices

12 April 2012 / Page 1 / 10.1126/science.1222360

V. Mourik,[1*] K. Zuo,[1*] S. M. Frolov,[1] S. R. Plissard,[2] E. P. A. M. Bakkers,[1,2] L. P. Kouwenhoven[1]†

[1]Kavli Institute of Nanoscience, Delft University of Technology, 2600 GA Delft, Netherlands.

[2]Department of Applied Physics, Eindhoven University of Technology, 5600 MB Eindhoven, Netherlands.



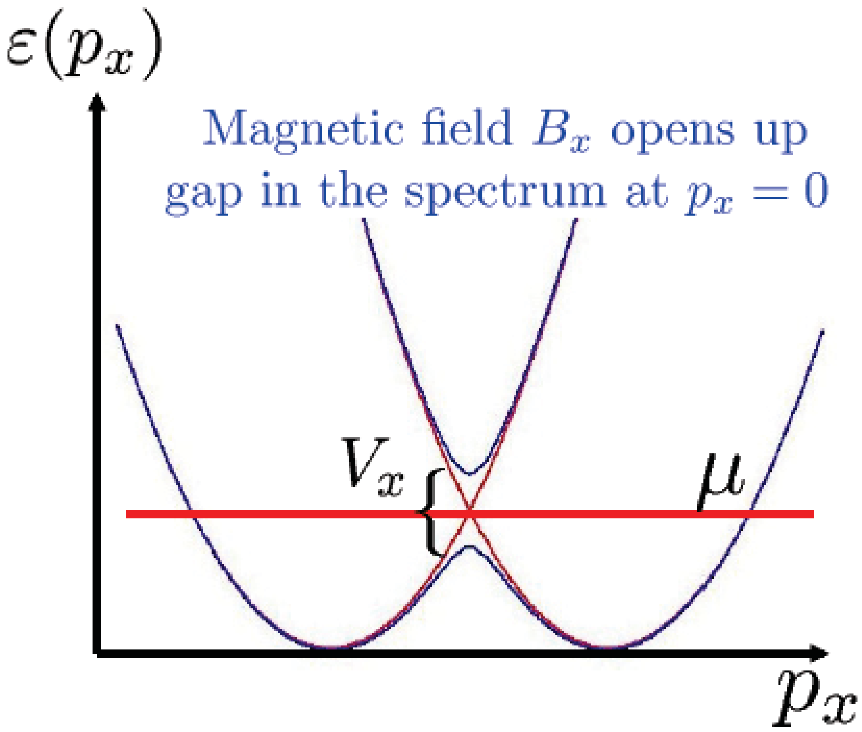
Coming back to the band structure

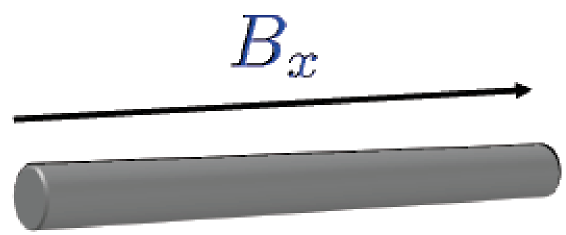
$$H_0 = \int_{-L}^{L} dx \psi_\sigma^\dagger(x) \left(-\frac{\partial_x^2}{2m^*} - \mu + i\alpha\sigma_y\partial_x + V_x\sigma_x \right)_{\sigma\sigma'} \psi_{\sigma'}(x)$$

single channel nanowire

spin-orbit coupling

Zeeman splitting





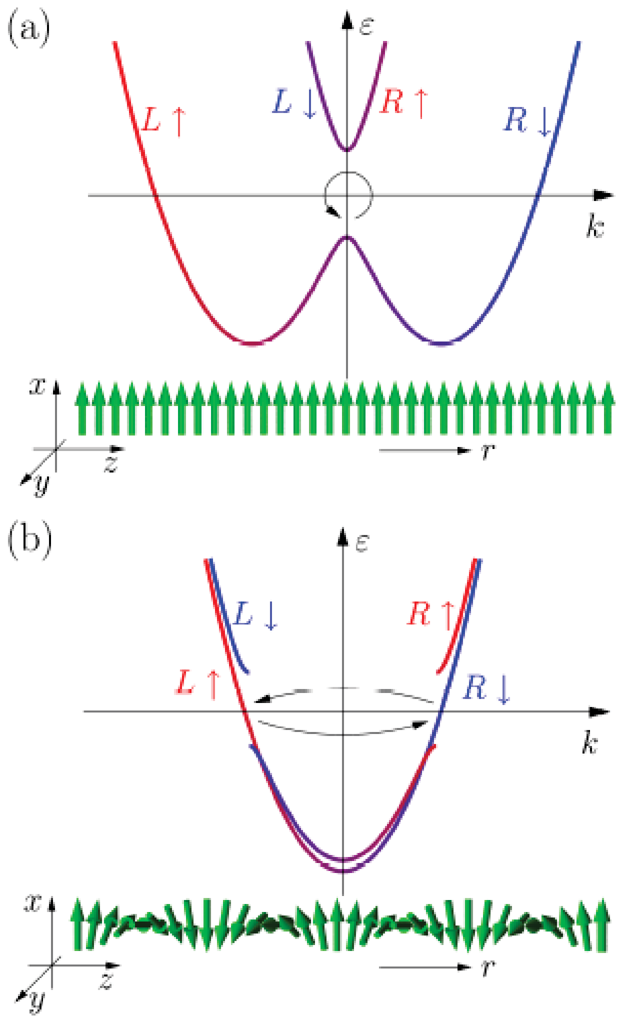
InAs, InSb nanowires

large spin-orbit ($\alpha \sim 0.1 eV \mathring{A}$)

large g-factor ($g \sim 10 - 50$)

good contacts with metals

Can we find other equivalent realizations ?



In 1D systems

Change of frame: Gauge transform

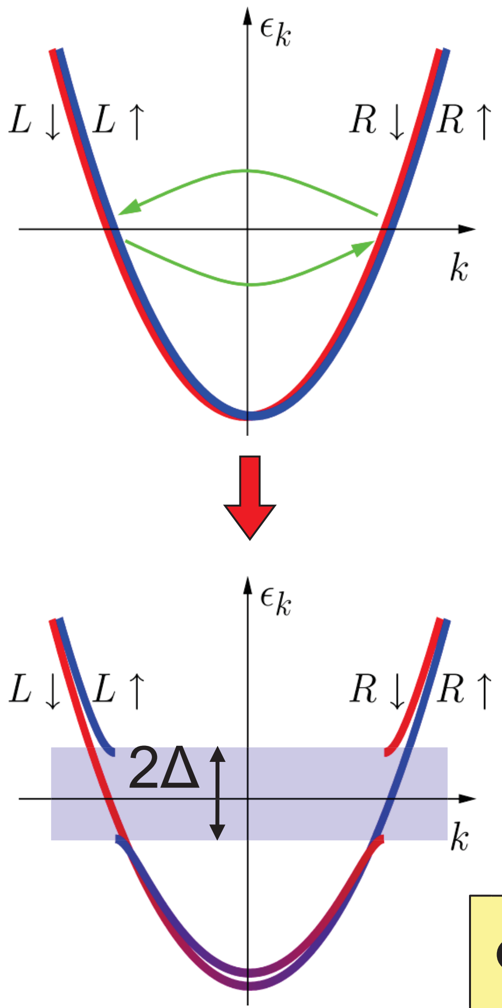
$$\psi_\sigma(r) \rightarrow e^{i\sigma k_{so} r}\psi_\sigma(r)$$

Equivalent to a helical magnetic field

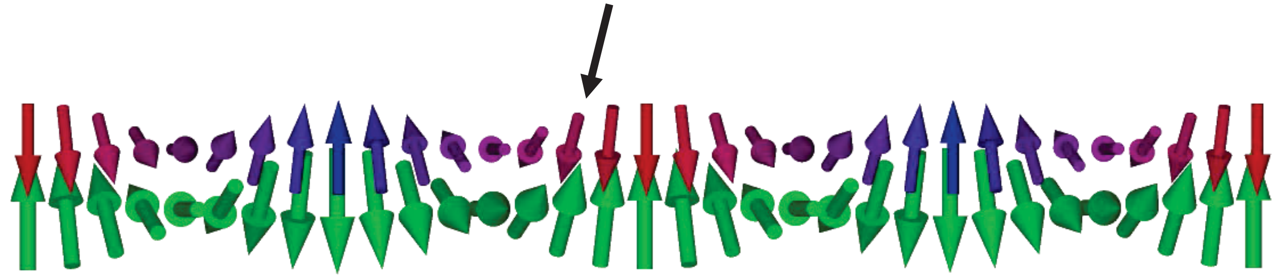
B. Braunecker et al. PRB 2010

The spin selective Peirls transition

external periodic potential: spiral magnetic field B(r)



backscattering on B(r): **gap Δ** for **one-half** of the conducting modes with **opposite spin** by forming a **helical spin/charge density wave**

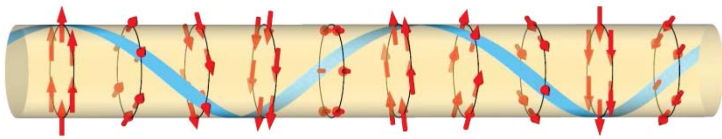


- **one-half** of the system becomes **insulating**
- **one-half** remains conducting and forms a **spin-filter**

B. Braunecker, G. Japaridze, J. Klinovaja, D. Loss, PRB 82, 045127 (2010)

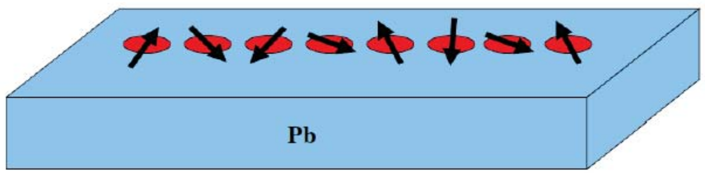
Where to find a magnetic helical field ?

➡ Quantum wires with magnetic moments (e.g. nuclear spins)



Braunecker, PS, Loss, PRL 2009, PRB 2009

➡ Magnetic nanoparticles on a SC without spin-orbit coupling



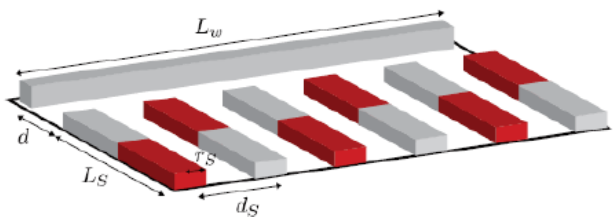
Choy et al., PRB 2011; Nakosai et al., PRB 2013

➡ In some rare earth materials where SC coexists with helimagnetism

e.g. $HoMo_6S_8$, $ErRh_4B_4$, and $TmNi_2B_2C$

Martin & Morpurgo, PRB 2012

➡ In superconducting nanowires without spin-orbit coupling



with spatially varying magnetic fields

Kjaergaard, Wolms, Flensberg, PRB 2012

Where to find a spiral field ?

Quantum wires with magnetic moments

Basic Hamiltonian

$$H = H_{\mathrm{el}} + \sum_i A\mathbf{S}_i^{el} \cdot \mathbf{I}_i^{nucl}$$

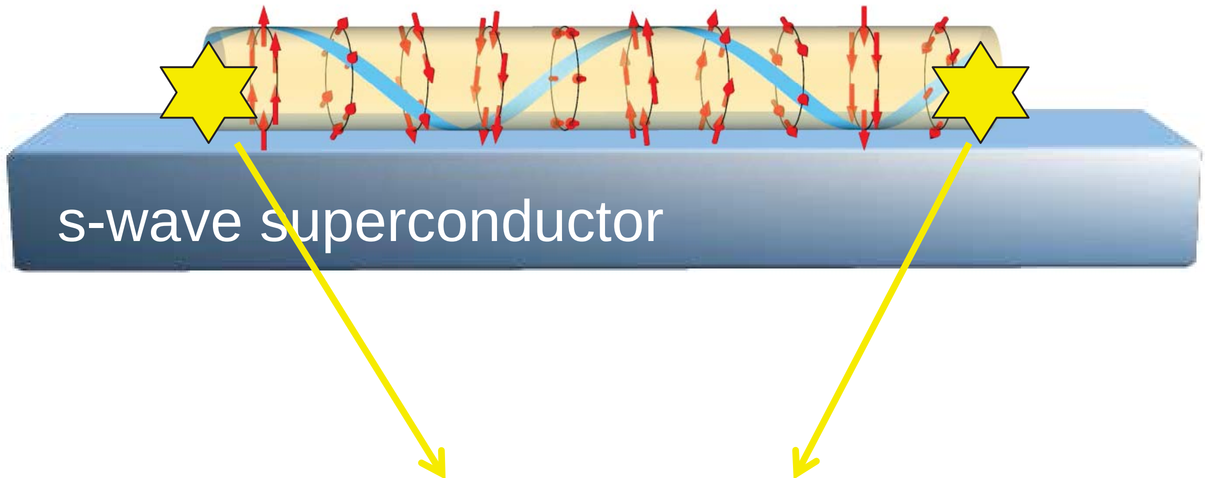
A 1D RKKY interaction is mediated between the magnetic moments by the conduction electron of the wire

Below some critical temperature T*, 1D electrons and magnetic moments can be tightly bound into an new ordered phase in 1D:
A 1D helical electron liquid+ helical magnetic phase

B. Braunecker, PS, D. Loss, PRL 2009, PRB 2009

Expect Majorana edge states ?

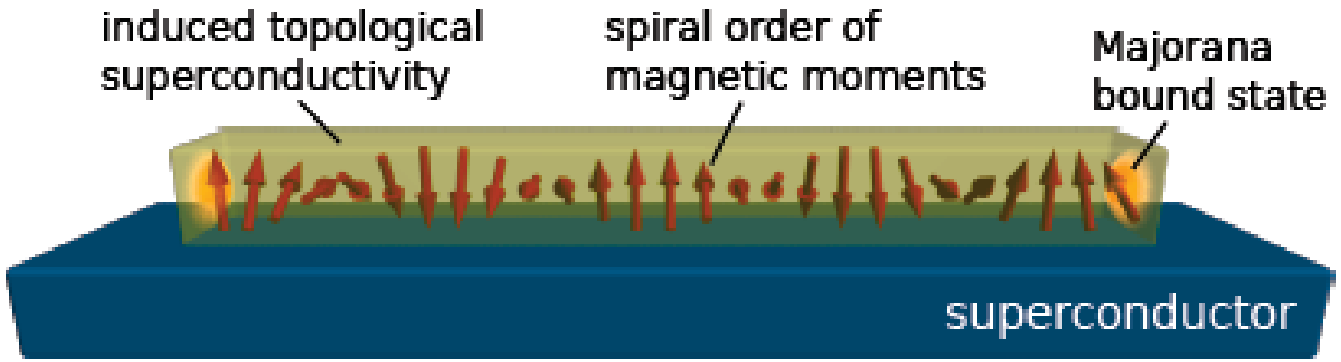
Superconductivity induced by proximity effect is of **p-wave type**
(projection onto spin-filtered conducting modes)



Majorana zero modes ?

S Gangadharaih, B Braunecker, PS, Loss, PRL 2011

Minimal ingredients and robustness



Recipe ingredients: - Magnetic moments interacting via 1D RKKY interactions

- 1D-like electronic band

- Proximized superconductivity

One can include also take electron-electron interactions into account:

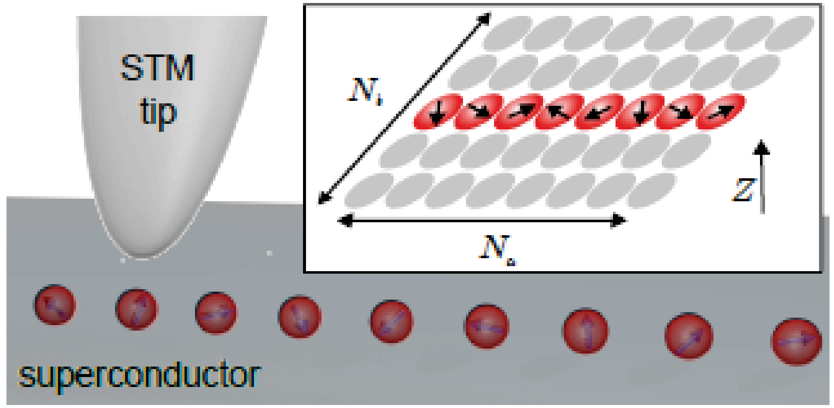
This enhances T^* but reduces the proximity induced gap (compromise)

Self-sustained topological superconducting phase without fine-tuning

B. Braunecker, PS, PRL (2013)
Vazifeh, M. Franz, PRL (2013)
Klinovaja, Stano, Yazdani, Loss, PRL (2013)

Does this locking scenario apply to magnetic adatoms on a SC substrate ?

Proposal: magnetic atoms with some **PRE EXISTING** spin texture on a SC surface



S. Nadj-Perge, I. Drozdov, A. Bernevig, A. Yazdani PRB (2013)

Recipe ingredients: for self-tuning

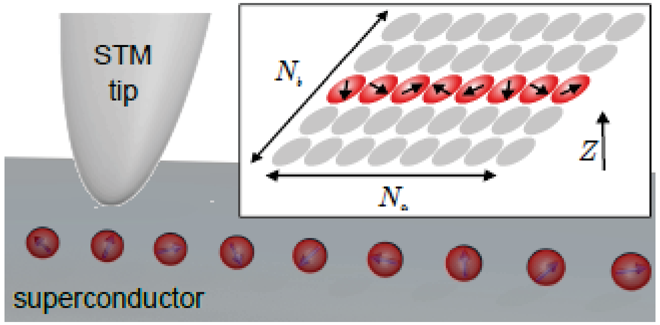
- Magnetic Classical moments with $\boxed{JS \ll E_F}$
- 1D-like electronic band that mediates RKKY interactions
- Proximized superconductivity

However, the substrate is **2D** here !
RKKY interaction in 2D favors ferromagnetic order
Helical ordering in 2D requires Dzyaloshinski-Morya interaction

Y. Kim et al., PRB (2014)

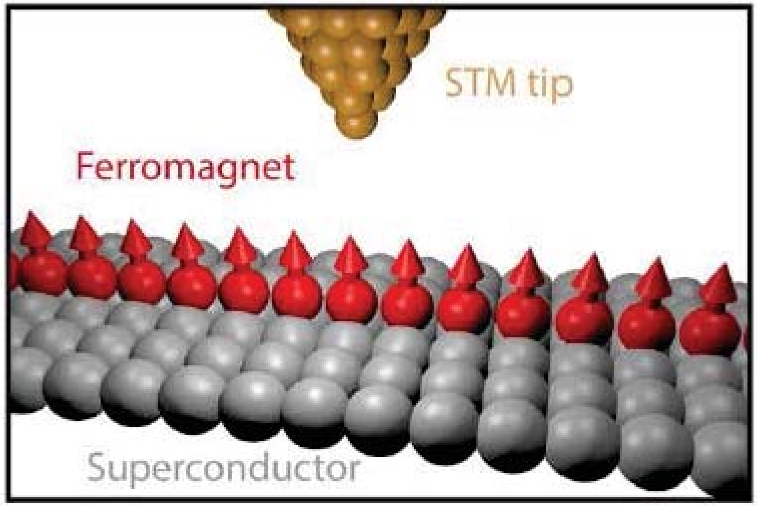
Proposed realizations for 1D topological SCs

- Chains of magnetic atoms on a superconductor



Nadj-Perge, Drozdov, Bernevig, Yazdani, PRB 2013

➡ Possible experimental realizations



Iron atoms on lead

Yazdani et al., Science 2014

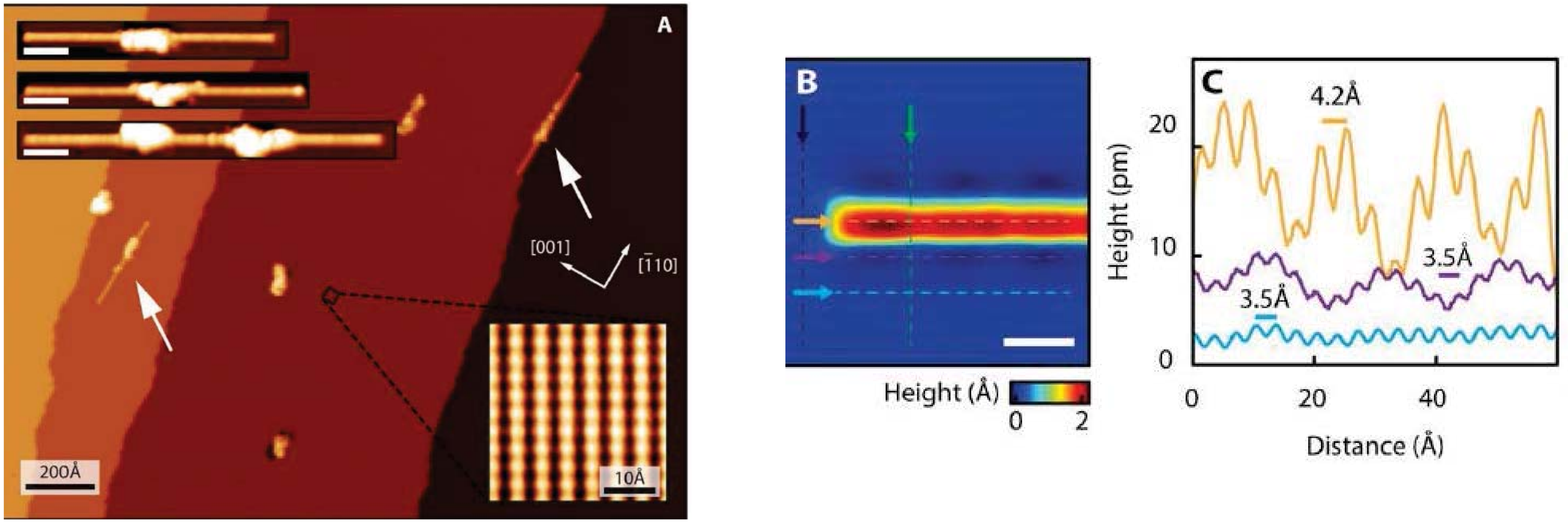
About the Princeton experiment

Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor

Stevan Nadj-Perge,[1*] Ilya K. Drozdov,[1*] Jian Li,[1*] Hua Chen,[2*] Sangjun Jeon,[1] Jungpil Seo,[1] Allan H. MacDonald,[2] B. Andrei Bernevig,[1] Ali Yazdani[1†]

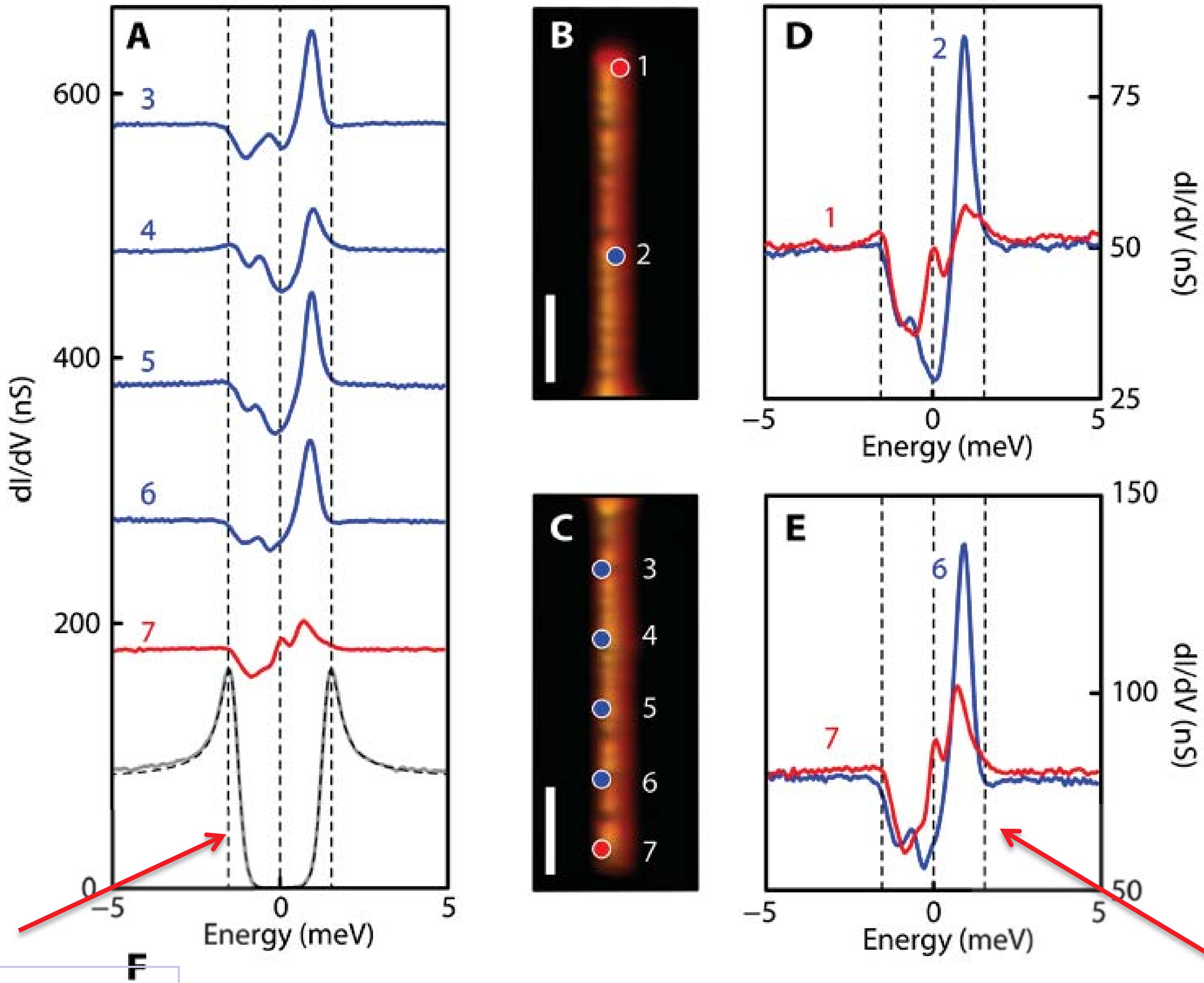
[1]Joseph Henry Laboratories and Department of Physics, Princeton University, Princeton, NJ 08544, USA.
[2]Department of Physics, University of Texas at Austin, Austin, TX 78712, USA.

Sciencexpress / http://www.sciencemag.org/content/early/recent / 2 October 2014



Ferromagnetic Fe atomic chains on the Pb(110) surface

About the Princeton experiment

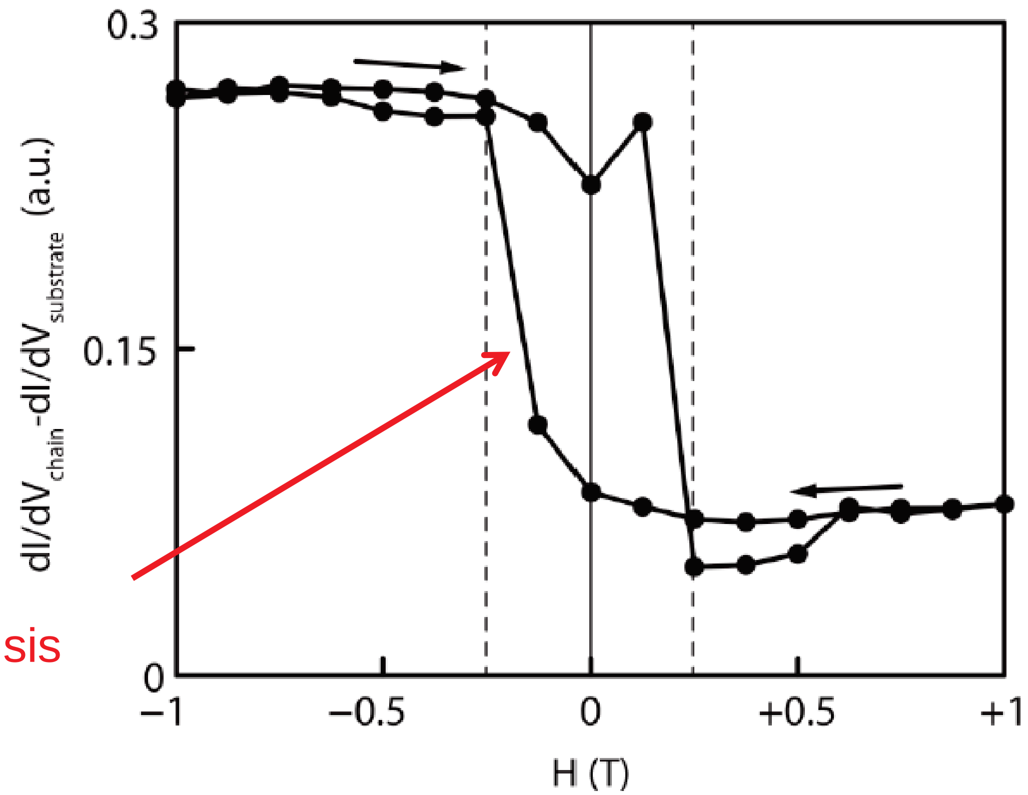


$\Delta_{Pb} \approx 1.36\text{meV}$

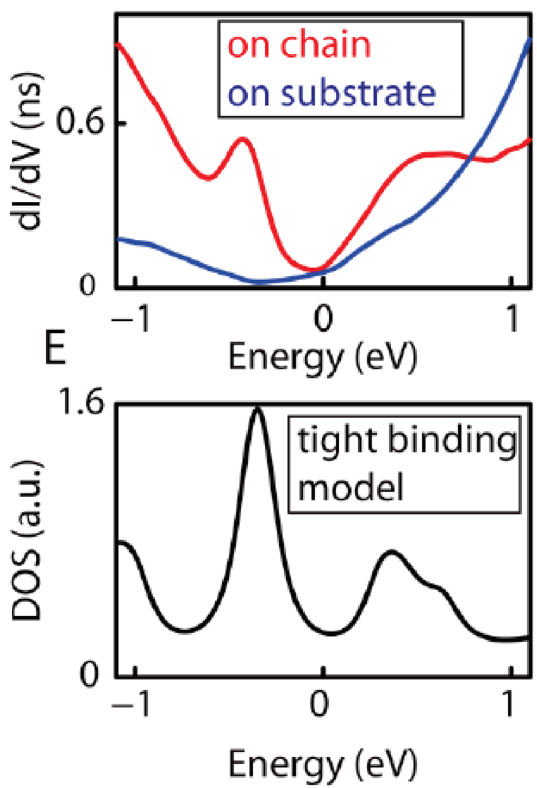
$\Delta_{\text{chain}} \approx 0.2-0.3\text{ meV}$

Spectroscopic mapping of atomic chains and ZBPs.

About the Princeton experiment



Hysterisis



Spin-polarized measurements with a Cr tip

Thèir analysis suggests ferromagnetism for the Fe iron chain with J around 2.4 eV

Pb has a strong Rashba spin-orbit coupling of order 100 meV

Back to the ``old" recipe like in semiconducting wires (SO+Zeeman+SC)

Majorana Spin polarization

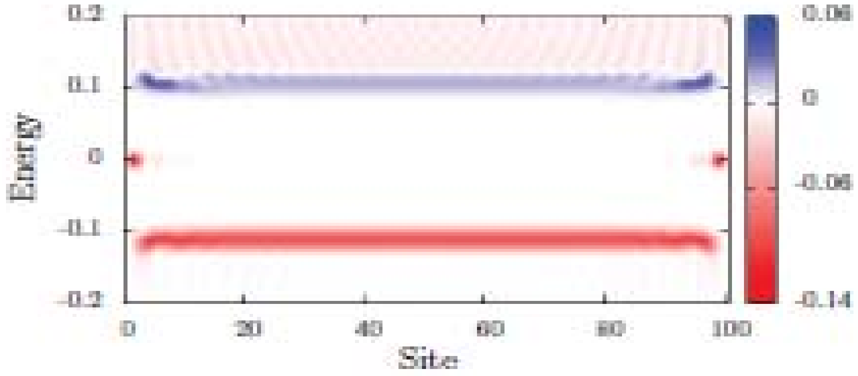
$$H = \sum_j \Psi_j^\dagger[(\mu - t)\tau_z + V_z\sigma_3 - \Delta\tau_1]\Psi_j - \frac{1}{2}\left[\Psi_j^\dagger(t + i\alpha\sigma_y + i\beta\sigma_x)\tau_z\Psi_{j+1} + \text{h.c.}\right]$$

Hamiltonian for topological semiconducting wires

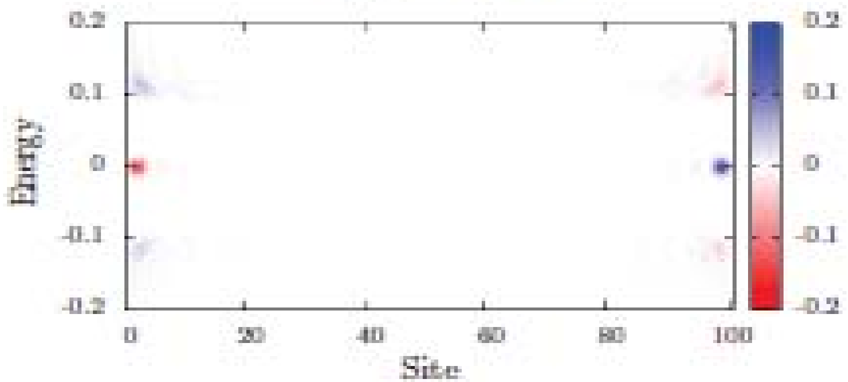
Condition for topological phase :

$$V_z^2 > \Delta^2 + \mu^2$$

z-axis spin polarization (electron component)



x-axis spin polarization (electron component)



D. Sticlet, C. Bena, PS, PRL (2012)

In the continuum limit

$$H = \int \Psi^\dagger \mathcal{H} \Psi \, dy, \quad \Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger, \psi_\downarrow, -\psi_\uparrow),$$

$$\mathcal{H} = \left(\frac{p^2}{2m} - \mu + \alpha p \sigma_y + \beta p \sigma_x \right) \tau_z + V_z \sigma_z - \Delta \tau_x.$$

$$E^2 = \xi^2 + (\alpha^2 + \beta^2) p^2 + V_z^2 + \Delta^2$$
$$\pm 2(\xi^2 (\alpha^2 + \beta^2) p^2 + \xi^2 V_z^2 + \Delta^2 V_z^2)^{1/2}$$

$\mu_0^2 > V_z^2 - \Delta^2$ $\mu_1^2 < V_z^2 - \Delta^2$ $\mu_0^2 > V_z^2 - \Delta^2$

One can find analytically the Majorana wave-function at zero energy

Electronic spin polarization

$$s(0) = \frac{|a|^2}{2}(-\sin(2\phi_1)\cos\vartheta, \sin(2\phi_1)\sin\vartheta, \cos(2\phi_1))$$

$$s(L) = \frac{|a|^2}{2}(\sin(2\phi_1)\cos\vartheta, -\sin(2\phi_1)\sin\vartheta, \cos(2\phi_1)).$$

$$e^{i\vartheta} = (\alpha + i\beta)/\sqrt{\alpha^2 + \beta^2}$$

$$e^{i\phi_j} = 1/\sqrt{2}(\sqrt{1-\mu_j/V_z} + i\sqrt{1+\mu_j/V_z})$$

In short, the electronic spin polarization in the (x,y) spin plane is always orthogonal to the SO vector $(\alpha, -\beta)/\sqrt{\alpha^2 + \beta^2}$

This electronic spin texture may potentially be probed experimentally

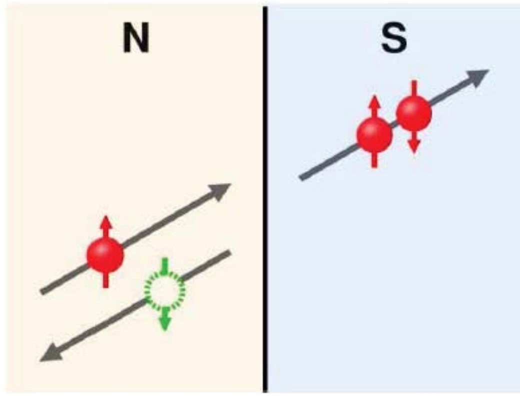
- by polarized STM   C Bena, D. Sticlet, PS, PRL (2012)

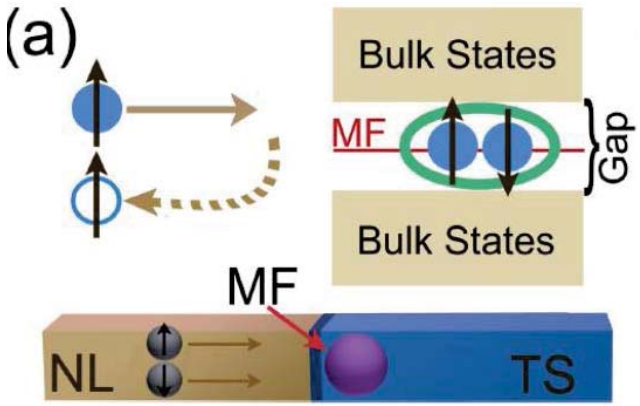
- by coupling the edge of the wire to a quantum dot   R. Zitko, PS, PRB (2012)

Spin-selective Andreev reflection   He et al., Phys. Rev. Lett. 112, 037001 (2014)

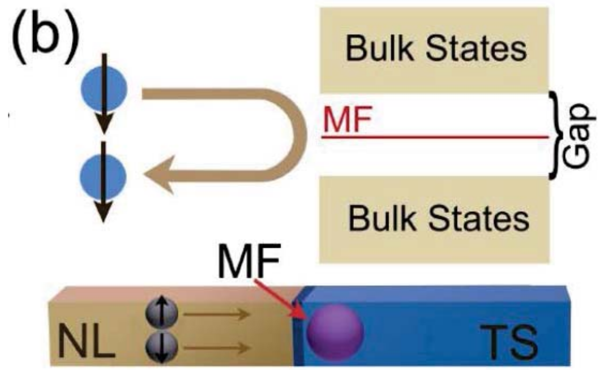
spin selective Andreev reflection

Standard Andreev reflection





Andreev reflected as a hole with the same spin



Perfect reflection

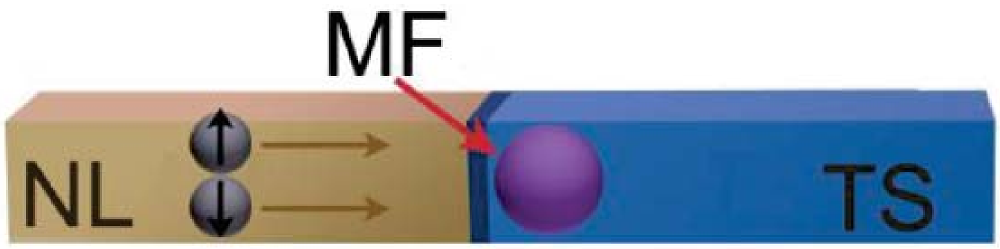
From He et al., Phys. Rev. Lett. 112, 037001 (2014)

Observation of Majorana fermions with spin selective Andreev reflection in the vortex of topological superconductor (Bi2Te3/NbSe2 hetero-structure)

H-H. Sun et al., Phys. Rev. Lett. 116, 257003 (2016)

spin selective Andreev reflection

Low energy description



$$H_T = H_L + H_c,$$
$$H_L = -iv_F \sum_{\alpha\in\uparrow/\downarrow} \int_{-\infty}^{+\infty} \psi_\alpha^\dagger(x)\partial_x\psi_\alpha(x)dx,$$
$$H_c = \tilde{t}\gamma[a\psi_\uparrow(0) + b\psi_\downarrow(0) - a^*\psi_\uparrow^\dagger(0) - b^*\psi_\downarrow^\dagger(0)]$$

Unitary transform

$$\Psi_1 = a\psi_\uparrow + b\psi_\downarrow \text{ and } \Psi_2 = -b^*\psi_\uparrow + a^*\psi_\downarrow$$
$$H_L = -iv_F \sum_{\alpha\in 1/2} \int_{-\infty}^{+\infty} \Psi_\alpha^\dagger(x)\partial_x\Psi_\alpha(x)dx,$$
$$H_c = \tilde{t}\gamma[\Psi_1(0) - \Psi_1^\dagger(0)].$$

Only one field couples to Majorana zero mode

Scattering matrix

$$\begin{pmatrix} \Psi_{1E}(+) \\ \Psi_{1E}^\dagger(+) \end{pmatrix} = \frac{1}{\Gamma + iE}\begin{pmatrix} iE & \Gamma \\ \Gamma & iE \end{pmatrix}\begin{pmatrix} \Psi_{1E}(-) \\ \Psi_{1E}^\dagger(-) \end{pmatrix} \qquad \Gamma = 2\tilde{t}^2/v_F$$

Andreev reflection amplitude: $\Gamma/(\Gamma + iE)$

From He et al., Phys. Rev. Lett. 112, 037001 (2014)

IIa) Physical realizations of 2D topological superconductors

Two-dimensional chiral p+ip superconductors

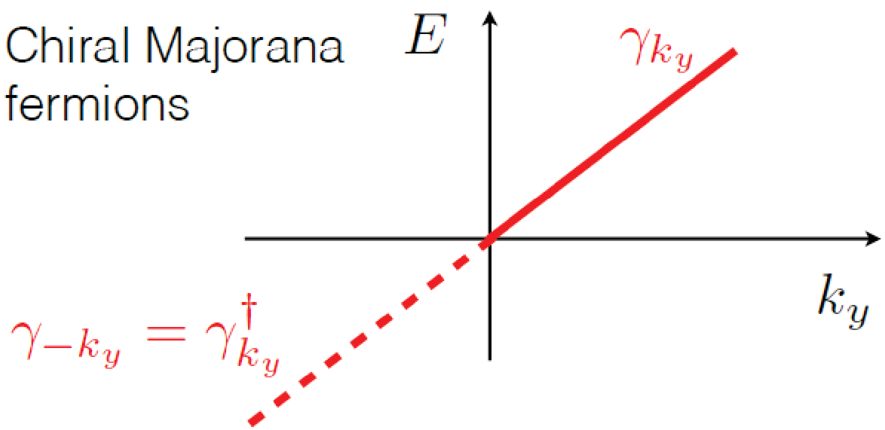
- The simplest nontrivial time-reversal breaking superconductor in 2D is the spinless p+ip superconductor

$$H_{\mathrm{BdG}} = \frac{1}{2}\sum_{\mathbf{p}} \Psi_{\mathbf{p}}^{\dagger} \begin{pmatrix} \frac{p^2}{2m} - \mu & 2i\Delta(p_x + ip_y) \\ -2i\Delta^*(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \Psi_{\mathbf{p}}$$

Weak (topological) pairing phase μ > 0

Diagonalize the Hamiltonian on the half plane x>0

Chiral Majorana fermions



$$H_{\mathrm{edge}} = \sum_{k_y \geq 0} v_F k_y \gamma_{-k_y} \gamma_{k_y}$$

Half of a chiral fermion !

Two-dimensional chiral p+ip superconductors

- Lattice BdG model:

$$\mathcal{H}_{\mathrm{BdG}} = (2t\,[\cos k_x + \cos k_y] - \mu)\,\tau_z + \Delta_0\,(\tau_x \sin k_x + \tau_y \sin k_y) = \boldsymbol{m}(\boldsymbol{k}) \cdot \boldsymbol{\tau}$$

$$E = \pm\,|\boldsymbol{m}(\boldsymbol{k})|$$

Spectrum flattening $\quad \hat{\boldsymbol{m}}(\boldsymbol{k}) = \dfrac{\boldsymbol{m}(\boldsymbol{k})}{|\boldsymbol{m}(\boldsymbol{k})|}$

classified by Chern number: (winding number) $\quad n = \dfrac{1}{8\pi}\displaystyle\int_{\mathrm{BZ}} d^2\mathbf{k}\,\epsilon^{\mu\nu}\,\hat{\mathbf{m}} \cdot \left[\partial_{k_\mu}\hat{\mathbf{m}} \times \partial_{k_\nu}\hat{\mathbf{m}}\right]$

Mapping: $\hat{\boldsymbol{m}}(\boldsymbol{k})$: Brillouin zone $\longmapsto$ $\hat{\boldsymbol{m}}(\boldsymbol{k}) \in S^2 \qquad$ "$\pi_2(S^2) = \mathbb{Z}$"

Two-dimensional chiral p+ip superconductors

- Intrinsic realizations of 2D p+ip superconductivity are scarce although there are a few important cases. They include:

1. the 5/2 fractional quantum Hall effect state that can be mapped onto a 2D p+ip superconductor (Read and Green, *"Paired states of fermions in two dimensions with breaking of parity and time reversal symmetries and the fractional quantum Hall effect"*, Phys. Rev. B, **61**, 10267 (2000).

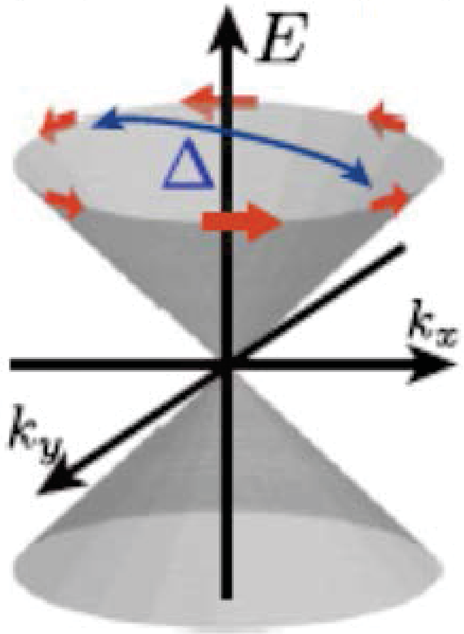
2. The intrinsic p+ip superconductor Sr2RuO4, see Mackenzie and Maeno, *"The superconductivity of Sr2RuO4 and the physics of spin triplet pairing"*, Rev. Mod. Phys. 75, 657, (2003); Das Sarma et al, *"Proposal to stabilize and detect halfquantum vortices in strontium ruthenate thin films: Non-Abelian braiding statistics of vortices in a px+ipy superconductor",* Phys. Rev. B,**73**, 220502 (2006); etc

Engineering 2D spinless p+ip topological superconductors

•One can also engineer systems that realize a topological phase supporting Majorana fermions in two dimensions by inducing an effective p+ip superconducting pairing in a spinless 2D electron gas.
Simplest case surface state of 3D TIs:

$$H_{3DTI} = \int d^2 r \psi^\dagger [-iv(\partial_x \sigma^y - \partial_y \sigma^x) - \mu] \psi$$

$$\epsilon_\pm(\mathbf{k}) = \pm v|\mathbf{k}| - \mu$$



•For any chemical potential residing within the bulk gap there us only one single Fermi surface (Dirac cones non-degenerate).

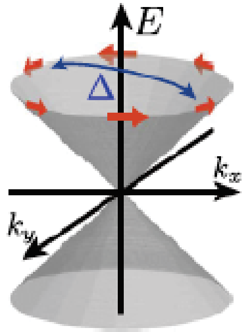
•Electrons along the Fermi surface are not spin-polarized (momentum-spin locking) so p+ip pairing can be effectively induced by s-wave proximity effect.

$$E_\pm(\mathbf{k}) = \sqrt{\epsilon_\pm^2(\mathbf{k}) + \Delta^2}$$

Engineering 2D spinless p+ip topological superconductors

$$H_{3DTI} = \int d^2r \psi^\dagger[-iv(\partial_x\sigma^y - \partial_y\sigma^x) - \mu]\psi$$

$$\epsilon_\pm(\mathbf{k}) = \pm v|\mathbf{k}| - \mu$$



$$H_\Delta = \int d^2r \Delta(\psi_\uparrow^\dagger\psi_\downarrow^\dagger + h.c)$$

$$H = \sum_{s=\pm} \int \frac{d^2\mathbf{k}}{2\pi} \{\epsilon_s(\mathbf{k})\psi_s^\dagger(\mathbf{k})\psi_s(\mathbf{k}) + [\frac{\Delta}{2}(\frac{k_x + ik_y}{|\mathbf{k}|})\psi_s(\mathbf{k})\psi_s(-\mathbf{k}) + h.c]\}$$

Time-reversal breaking of any form will generate chiral Majorana edge states at the boundary between topologically superconducting and magnetically gapped regions in the surface of a 3D TI.

Fu and Kane, PRL 100, 096407, 2008

Engineering 2D spinless p+ip topological superconductors

$$H = \sum_{s=\pm} \int \frac{d^2\mathbf{k}}{2\pi} \{\epsilon_s(\mathbf{k})\psi_s^\dagger(\mathbf{k})\psi_s(\mathbf{k}) + [\frac{\Delta}{2}(\frac{k_x + ik_y}{|\mathbf{k}|})\psi_s(\mathbf{k})\psi_s(-\mathbf{k}) + h.c]\}$$

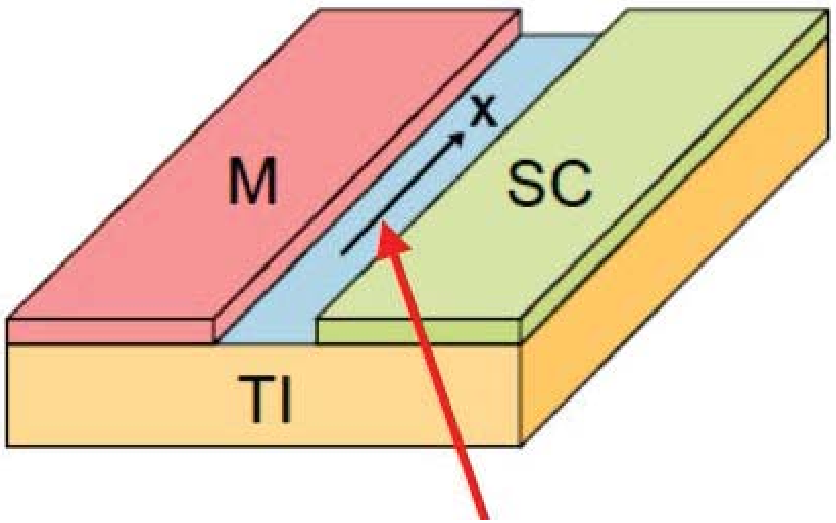
$$H_Z = -h \int d^2r \psi^\dagger \sigma^z \psi$$

$$H_{edge} = -i\hbar v_F \gamma \partial_x \gamma$$

$$\epsilon_\pm(\mathbf{k}) = \pm v|\mathbf{k}| - \mu$$

$$\Downarrow$$

$$\epsilon_\pm(\mathbf{k}) = \pm\sqrt{v^2\mathbf{k}^2 + h^2} - \mu$$

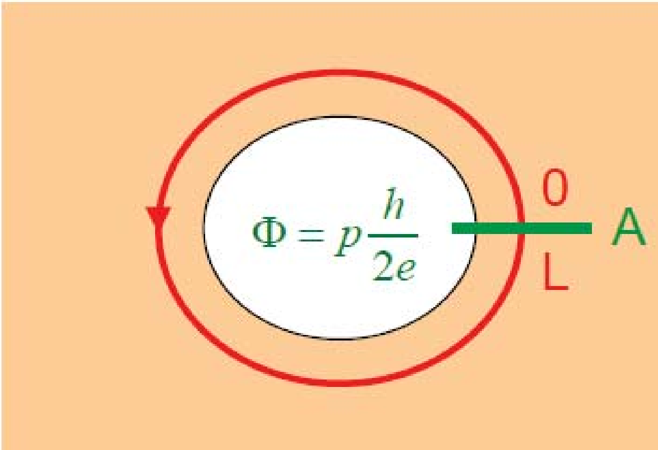


Time-reversal breaking of any form will generate **chiral Majorana edge states** at the boundary between topologically superconducting and magnetically gapped regions in the surface of a 3D TI.

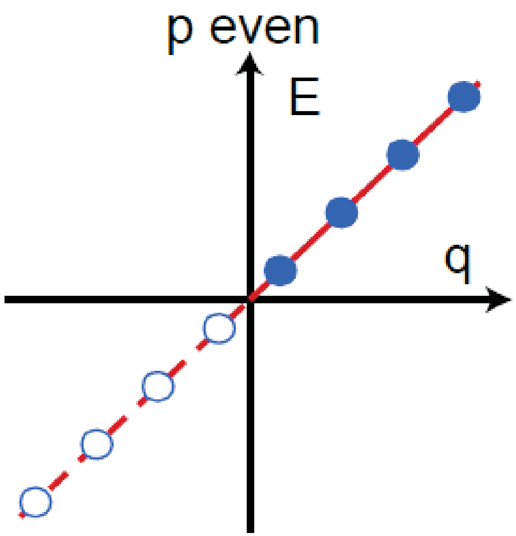
Fu and Kane, PRL 100, 096407, 2008

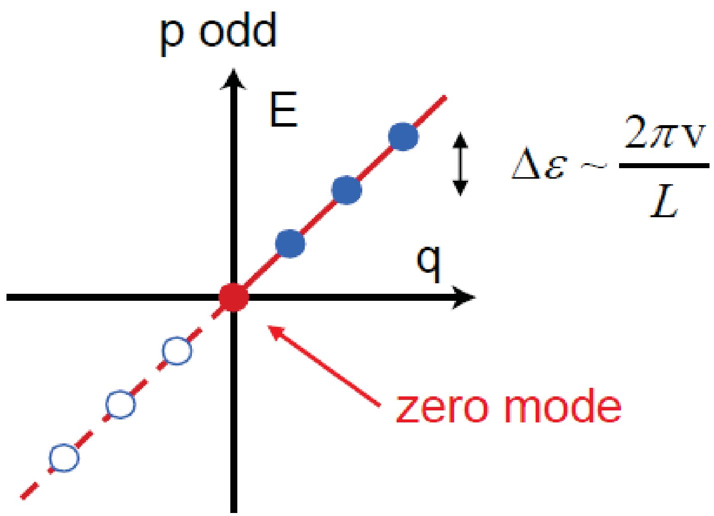
Majorana zero mode at a vortex

Consider a **hole** in a p-wave superconductor threaded by a magnetic flux



$$\psi(L) = (-1)^{p+1}\psi(0)$$

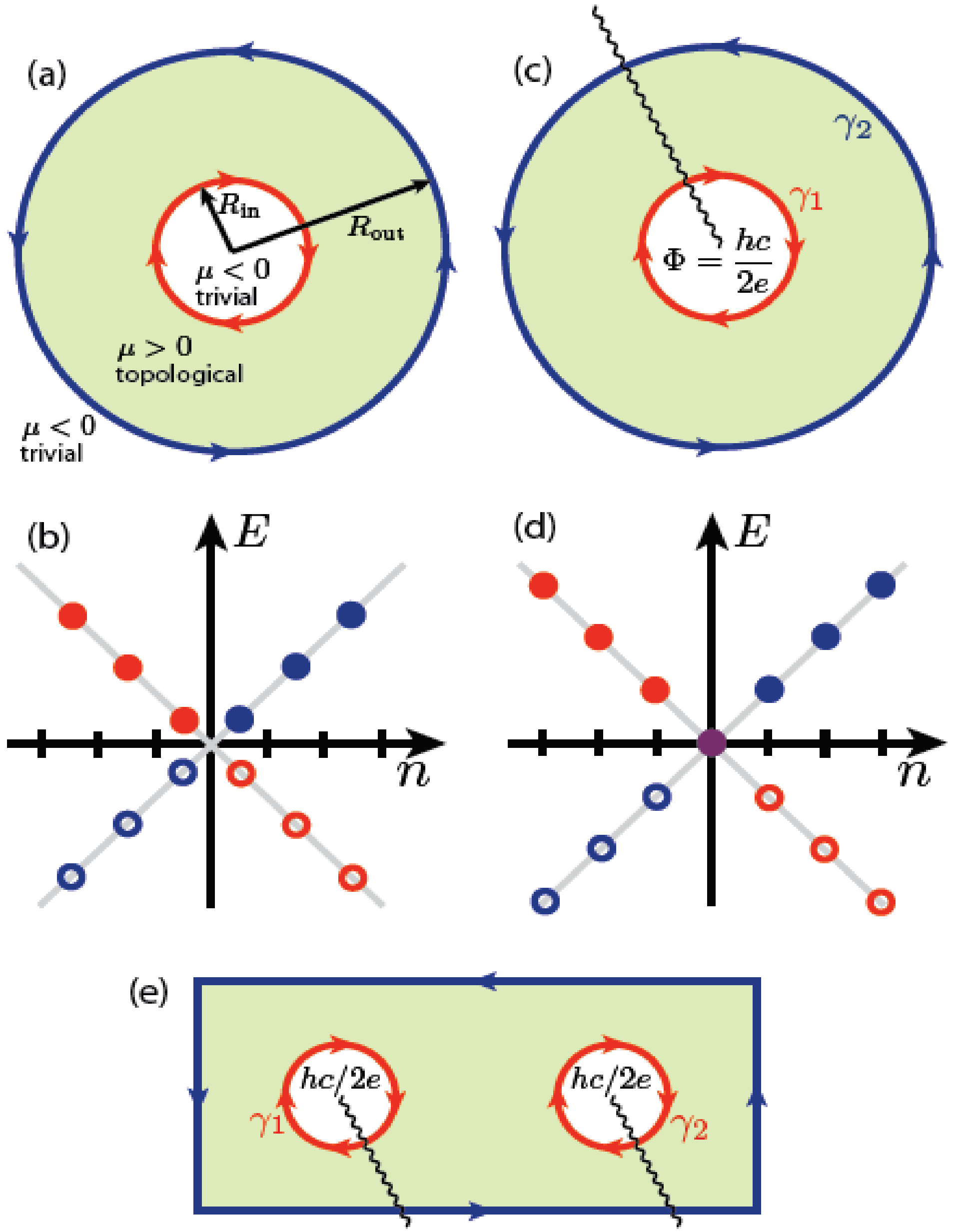




Without the hole : $\delta\epsilon = \dfrac{\Delta^2}{E_F}$

Caroli, de Gennes, Matricon theory ("64)

Majorana zero mode at a vortex



From Alicea, Rep.Prog. Phys. 75, 076501 (2012)